Probability theory (basics)

In probability, an experiment or trial is any procedure that has a well-defined set of possible outcomes. The set of all possible (relevant) outcomes is the **sample space** of the experiment. Usually denoted by Ω or S.

Any subset of the sample space is called an **event**. ("elementary" approach, works well for finitely many outcomes)

P1 Describe and write down the sample space for an experiment in which two coins are thrown and each shows either a head or a tail. Describe some of the events of the experiment.

Sol: If we let H denote the occurrence of a head and T the occurrence of a tail, then ~~the~~ we can represent the sample space of the experiment as the following set of ordered pairs:

$$\Omega = \{(H,H), (H,T), (T,H), (T,T)\}.$$

The outcome (H, H) will be if both coins are heads, (H, T) if the first coin is heads and the second coin is tails, etc.

Event $E_1 = \{(H,H),(H,T)\}$ occurs if the first coin is heads and the second is heads, or if the first is heads and the second is tails, hence E is the event that a head appears on the first coin. Similarly $E_2 = \{(H,H),(T,H)\}$ is the event that a head appears on the secant coin, etc.

Probability is any function P from the set of events to the set of real numbers satisfying the following three axioms:

(I) $\quad 0 \le P(E)$ and $P(E) \le 1$ for all events E.

(II) $\quad P(\Omega) = 1$ (i.e. probability of the whole sample space is 1)

(III) For any sequence of mutually exclusive events $E_1, E_2, \dots$ (i.e. events such that $E_i \cap E_j = \emptyset$ if $i \neq j$)

$$P\left(\bigcup_{i=1}^{+\infty} E_i\right) = \sum_{i=1}^{+\infty} P(E_i)$$

P2 Suppose a die is rolled and all sides are equally likely. Using the probability axioms compute the probability of rolling an odd number.

$\boxed{\text{Sol:}}$ Our sample space is $\Omega = \{1,2,3,4,5,6\}$. Since all sides are equally likely:

$$P(\{1\}) = P(\{2\}) = \dots = P(\{6\}). \quad (*)$$

From the axiom (II) and (III) we have for $i \in \{1,\dots,6\}$:

$$1 \overset{(II)}{=} P(\Omega) \overset{(III)}{=} P(\{1\}) + P(\{2\}) + \dots + P(\{6\}) \overset{(*)}{=} 6\,P(\{i\}),$$

hence $P(\{i\}) = \frac{1}{6}$, for all $i \in \{1, \dots, 6\}$. Using the axiom (III) now we have:

$$P(\{1,3,5\}) \overset{(III)}{=} P(\{1\}) + P(\{3\}) + P(\{5\}) = 3 \cdot \frac{1}{6} = \frac{1}{2},$$

so the probability of rolling an odd number is $\frac{1}{2}$.

A generalization of P2 is an experiment in which we have finitely many (say N) equally likely outcomes. Then we can write our sample space as $\Omega = \{1, 2, \dots, N\}$, and as in P2 deduce that

$P(\{i\}) = \frac{1}{N}$ for all $i \in \Omega$, and that for any event

E we have:

$$P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } \Omega \; (= N)}$$

P3 If two dice are rolled, what is the probability that the sum of the upturned faces will equal 7?

$\boxed{\text{Sol:}}$ Assuming that the dice are fair, our sample space will be the set of 36 ordered pairs:

$$\Omega = \{(1,1), (1,2), \dots, (6,6)\},$$

and all the outcomes are equally likely.

Our event of rolling the sum 7, denote it by E_7 is

$$E_7 = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}.$$

and so $$P(E_7) = \frac{|E_7|}{|\Omega|} = \frac{6}{36} = \frac{1}{6}.$$

P4 A committee of 5 is to be selected from a group of 6 men and 9 women. If the selection is made randomly, what is the probability that the committee consists of 3 men and 2 women?

$\boxed{\text{Sol:}}$ Our sample space consists of sets of 5 people chosen randomly from a set of 15 people, hence $|\Omega| = \binom{15}{5}$. Let E denote the event: chosen committee has 3 men and 2 women. There are $\binom{6}{3}$ way to choose the 3 men and $\binom{9}{2}$ to choose the 2 women, hence $|E| = \binom{6}{3}\binom{9}{2}$, and so

$$P(E) = \frac{|E|}{|\Omega|} = \frac{\binom{6}{3}\binom{9}{2}}{\binom{15}{5}} = \frac{\frac{6\cdot5\cdot4}{1\cdot2\cdot3}\cdot\frac{9\cdot8}{1\cdot2}}{\frac{15\cdot14\cdot13\cdot12\cdot11}{1\cdot2\cdot3\cdot4\cdot5}} = \frac{20\cdot36}{7\cdot13\cdot3\cdot11} = \frac{240}{1001}$$

P5 If 3 balls are randomly drawn from a bowl containing 6 white and 5 red balls, what is the probability that one of them is white and the other two red?

$\boxed{\text{Sol:}}$ Assume that we are drawing the balls one after the other. We have 11 options for the first one, 10 for the second and 9 for the third, so our sample space has $11\cdot10\cdot9$ elements. Let E be the event:

one ball is white and two are red. If white is drawn first then we have 6 options for it and 5 options for red on the second drawn and 4 for the red on the third, hence in this case we have $6 \cdot 5 \cdot 4$ options. Similarly if white is drawn second and third, hence $|E| = 3 \cdot (6 \cdot 5 \cdot 4)$, and so

$$P(E) = \frac{|E|}{|\Omega|} = \frac{3 \cdot 6 \cdot 5 \cdot 4}{11 \cdot 10 \cdot 9} = \frac{4}{11}.$$

Conditional probability is a measure of the probability of an event occurring, given that another event has already occurred.

Given two event E and F such that $P(F) > 0$, the conditional probability that E occurs given that F has occurred is defined as

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

P6 A fair coin is flipped twice. What is the probability that both flips land on heads, given that:
a) the first flip lands on heads;
b) at least one flip lands on heads?

Sol: Our sample space is $\Omega = \{(H,H),(H,T),(T,H),(T,T)\}$. Let $A = \{(H,H)\}$ denote the event that both flips land on heads; $B = \{(H,H),(H,T)\}$ the event that heads is obtained on the first flip, and $C = \{(H,H),(H,T),(T,H)\}$

be the event that at least one heads is obtained.
Then we have:

a)
$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\{H,H\})}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

b)
$$P(A \mid C) = \frac{P(A \cap C)}{P(C)} = \frac{P(\{(H,H)\})}{P(\{(H,H)\}) + P(\{(H,T)\}) + P(\{(T,H)\})}$$

$$= \frac{\frac{1}{4}}{3 \cdot \frac{1}{4}} = \frac{1}{3}.$$

P7 Celine estimates that her probability of receiving an A grade in an English course is $\frac{1}{2}$ and $\frac{2}{3}$ in a chemistry course. If Celine decides to base her decision on the flip of a coin, what is the probability that she gets an A in chemistry?

$\boxed{\text{Sol:}}$ Let C be the event that the coin chooses chemistry, and E it chooses English. Let F be the event: Celine gets an A. Then the event: Celine gets an A in chemistry is $C \cap F$, and we have

$$P(C \cap F) = P(F \mid C) \cdot P(C) = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}.$$

Bayes' theorem: Let A and B be events, then the following formula holds:

$$P(A|B) = \frac{P(B|A)\cdot P(A)}{P(B|A)\cdot P(A) + P(B|A^c)\cdot P(A^c)}.$$

(Here A^c is the complement of A in Ω, i.e. $A^c = \Omega \setminus A$.)

P8 A group of police officers have breathalyzers displaying false drunkenness in 5% of the cases in which the driver is sober. However, the breathalyzers never fail to detect a truly drunk person. One in a thousand drivers is driving drunk. Suppose the police officers then stop a driver at random to administer a breathalyzer test. It indicates that the driver is drunk. We assume you do not know anything else about them. How high is the probability that they really are drunk?

Sol: Let D be the event that the driver is drunk. and let A be the event that the test is positive.

From the statement of the problem we have:

$$P(D^c) = \frac{999}{1000}$$

$$P(D) = \frac{1}{1000} \; ; \quad P(A|D) = 1 \; ; \quad P(A|D^c) = \frac{5}{100} = \frac{1}{20}.$$

So, using Bayes' theorem:

$$P(D|A) = \frac{P(A|D)\cdot P(D)}{P(A|D)\cdot P(D) + P(A|D^c)\cdot P(D^c)} = \frac{1\cdot\frac{1}{1000}}{1\cdot\frac{1}{1000} + \frac{1}{20}\cdot\frac{999}{1000}}$$

$$= \frac{20}{1019} = 0{,}019627\ldots$$

P9 At a certain stage of investigation, the inspector is 60% convicted of the guilt of a certain suspect. A new piece of evidence is uncovered which shows that the criminal is left-handed. Suppose that 20% of population is left-handed. If the suspect is left-handed, how certain should the inspector be now that he is the criminal?

Sol: Let G be the event that the suspect is guilty, and L that the criminal is left-handed. From the statement of the problem we have:

$$P(G) = 0.6 \qquad P(L) = P(L|G^c) = 0.2 \text{ and } \qquad P(L|G) = 1$$

$\Downarrow$

$P(G^c) = 1 - 0.6 = 0.4$; So, using Bayes' formula:

$$P(G|L) = \frac{P(L|G)\cdot P(G)}{P(L|G)\cdot P(G) + P(L|G^c)\cdot P(G^c)} =$$

$$= \frac{1\cdot 0.6}{1\cdot 0.6 + 0.2\cdot 0.4} = \frac{0.6}{0.68} = 0.882$$

P10 Suppose that we have 3 cards identical in form. Both sides of the first one are blue, both sides of the second one are red, and one side of the third card is blue and the other side is red. One card is randomly selected and put down on the ground. If the upper side of the card is red, what is the probability that the other side is blue?

Sol: Let RR, BB, RB respectively denote the events: chosen card is all red, it is all blue, and it is red-blue. Let R be the event: the upturned side of the card is red.

Then:

$$P(R) = P(R \cap (RR \cup BB \cup RB)) = P((R \cap RR) \cup (R \cap BB) \cup (R \cap RB))$$

$$= P(R \cap RR) + P(R \cap BB) + P(R \cap RB) =$$

$$= P(R|RR) \cdot P(RR) + P(R|BB) \cdot P(BB) + P(R|RB) \cdot P(RB)$$

$$= 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3}$$

$$= \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

So, from the definition of conditional probability:

$$P(RB|R) = \frac{P(R \cap RB)}{P(R)} = \frac{P(R|RB) \cdot P(RB)}{P(R)} = \frac{\cancel{\frac{1}{2}} \cdot \frac{1}{3}}{\cancel{\frac{1}{2}}} = \frac{1}{3}$$

Hence the probability that the other side is blue is $\frac{1}{3}$.