Teoretične osnove računalništva 2
Teorija informacij

Channels with additive Gaussian noise

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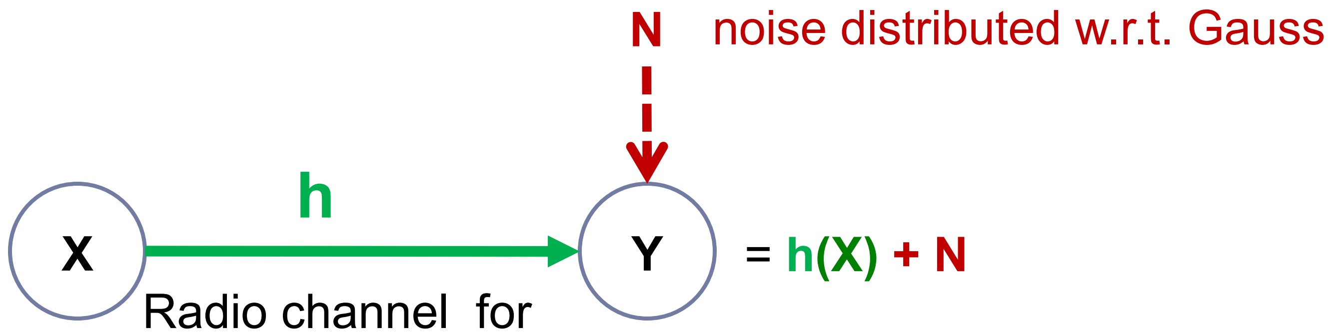
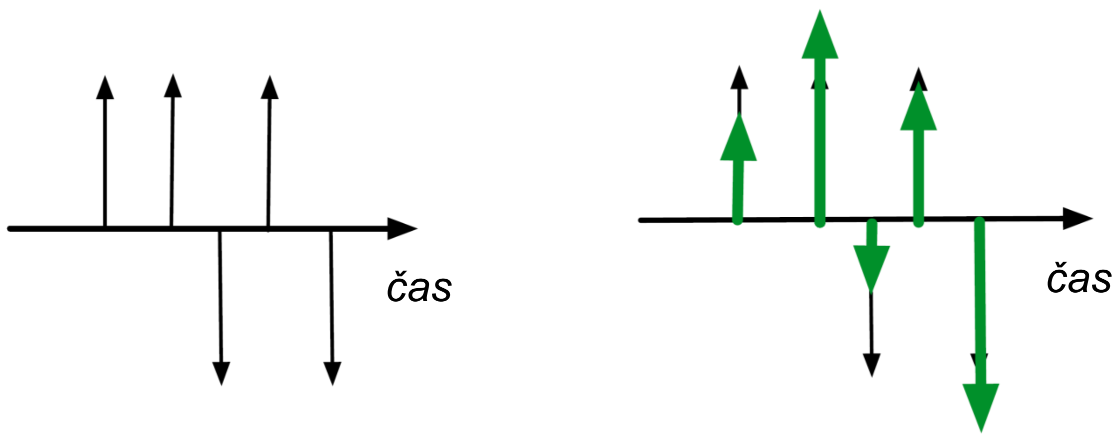
Lecture overview

- Gaussian model of communication channel:
  - definition, channel capacity,
  - Frequency limited channels:
    - Nyquist sampling theorem and
    - capacity of frequncy limited channels
  - capacity of frequency non-limited channels

Information processing over communication channels

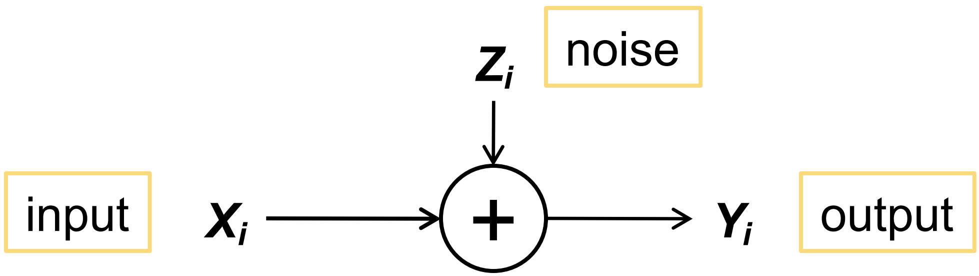


Radio channel for
wireless communication
Rayleigh's channel



Gaussian model of communication channel

- Gaussian model of communication channel:

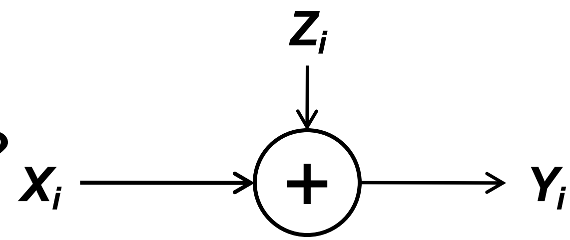


$$Y_i = X_i + Z_i, \qquad Z_i \sim \mathcal{N}(0, N).$$

Z_i is **INDEPENDENT** *of* X_i.

- There are numerous communication channels of this kind:
  - Wired and wireless telefon connections,
  - Satelite connections, ...

Gaussian model of communication channel



- **Is it possible that the capacity is infinite ?** **YES,** if no restrictions.

- If the **noise variance of is 0**, we have ERROR-FREE transmission. Since X can take **any real value**, the iput alphabe can be infinite. **The capacity of such channel is then infinite.**

- If the **noise variance of is NOT 0 and the input values can take any real value** (not limited in power), the **kapaciteta of channel is infinite.**

  IDEA: Distinguish input values properly so that the noise “cannot” affect the signals (no errors)

Gaussian model of communication channel

- NONE of the previous scenario is realistic !

- Commonly, the input is power limited and cannot have infinite energy (power).
- This is moddeled as below so that avarage power of symbols x_1, x_2, . . . , x_n (each having a power x_i^2)

Za poljuben niz simbolov $(x_1, x_2, \ldots, x_n)$, ki jih lahko prenašamo po kanalu velja omejitev:

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 \leq P.$$

- Example: radio and satelite connections

Model of communication channel: Gaussian noise

- The noise in a communication channels is caused by many different sources of noise.
- Central limit theorem states:
  The sum of many random noise contributions behaves approximately as being distributed „normally“- $N\,(0,\sigma^2)$
- **Therefore, we assume that the noise is a continuous random variable with normal distribution and the mean value 0 with non-zero variance** $N\,(0,\sigma^2)$

Definition of time –discrete Gaussian kommunication channel

Gaussian time-discrete channel

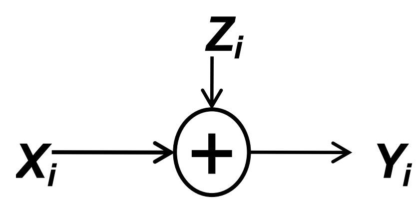
Additive Gaussian model of communication channel is **time-discrete channel** which at time i has input X_i, noise Z_i and output Y_i so that

$$Y_i = X_i + Z_i,$$

where Z_i is independent of X_i (also Z_i independent from Z_j) and normally distributed $\sim \mathcal{N}(0, \sigma^2 = N)$.

- We have condition on the **average power** of input signal :

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 \leq P \Rightarrow E[X^2] \leq P.$$



$$E[Y^2] = E[(X+Z)^2] = E[X^2] + 2E[X]E[Z] + E[Y^2] \leq P + N.$$

Example of using Gaussian communication channel

We send 1 bit of information through the channel. Assuming we are transmitting signals which are power limited the following coding can be used: $+\sqrt{P}$ for "1" and $-\sqrt{P}$ for "0".

Receiver gets distorted versions Y and tries to decide what has been sent. An optimal approach is to vote for "1" if $Y > 0$ and for "0" if $Y < 0$. The bit error probability for such a system is:

$$
\begin{aligned}
P_e &= \frac{1}{2}\Pr(Y < 0|X = +\sqrt{P}) + \frac{1}{2}\Pr(Y > 0|X = -\sqrt{P}) \\
&= \frac{1}{2}\Pr(Z < -\sqrt{P}|X = +\sqrt{P}) + \frac{1}{2}\Pr(Z > \sqrt{P}|X = -\sqrt{P}) \\
&= \Pr(Z > \sqrt{P}) \\
&= 1 - \Phi\left(\sqrt{P/N}\right),
\end{aligned}
$$

Cumulative function for normal distribution

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}}\, dt$$

Binary symmetric channel from Gaussian channel

- We have represented Gaussian channel as binary simmetric channel with probability of error P_e.
- Similarly, we can translate Gaussian channel with 4 input signals into a discrete channel with 4 input symbols.
- This is how some modulation schemes work in practice, thus **transforming continuous channel into a discrete one.** Then, we use the knowledge about discrete channels.
- Some information will be lost because of quantization.

Probability mass distribution- reminder

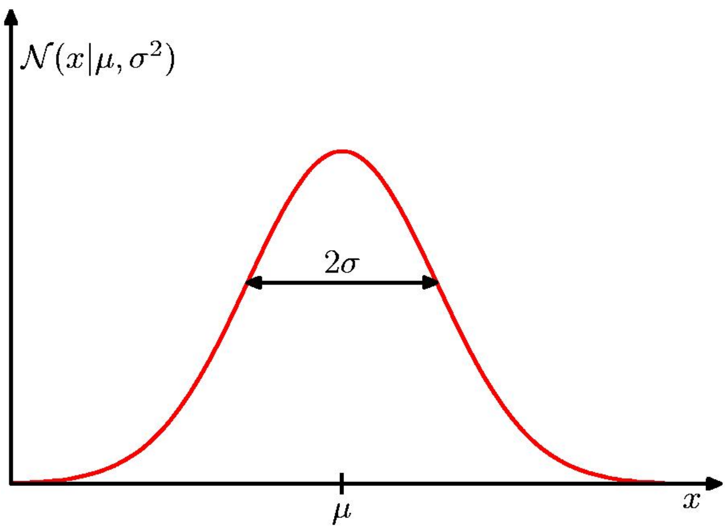
Probability mass distribution

Let X be a random variable with the **cumulative probability function** $F(x) = Pr(X < x)$. If $F(x)$ is continuous then X is called continuous RV. With $f(x)$ we denote derivative of F, i.e. $f(x) = F'(x)$.

If we have $\int_{-\infty}^{+\infty} f(x)dx = 1$, then $f(x)$ is called probability mass distribution of X.

Normal (Gaussian) distribution

$$\mathcal{N}(x|\mu,\sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$



Diferential entropy - reminder

Entropija zvezne naključne spremenljivke

Entropija zvezne naključne spremenljivke X s porazdelitvijo gostote verjetnosti $f(x)$ je definirana kot:

$$h(X) = -\int_S f(x) \log f(x) dx,$$

kjer je S nosilec naključne spremenljivke X.

Differential entropy for normal distribution - reminder

- Probability mass distribution *X ~ N(0, σ)*:

$$X \sim \phi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-x^2}{2\sigma^2}}$$

- Entropy:

$$\begin{aligned} h(\phi) &= -\int \phi \ln \phi \\ &= -\int \phi(x)\left[-\frac{x^2}{2\sigma^2} - \ln\sqrt{2\pi\sigma^2}\right] \\ &= \frac{EX^2}{2\sigma^2} + \frac{1}{2}\ln 2\pi\sigma^2 \\ &= \frac{1}{2} + \frac{1}{2}\ln 2\pi\sigma^2 \\ &= \frac{1}{2}\ln e + \frac{1}{2}\ln 2\pi\sigma^2 \\ &= \frac{1}{2}\ln 2\pi e\sigma^2 \quad \text{nats.} \end{aligned}$$

$$h(\phi) = \frac{1}{2}\log 2\pi e\sigma^2 \quad \text{bits.}$$

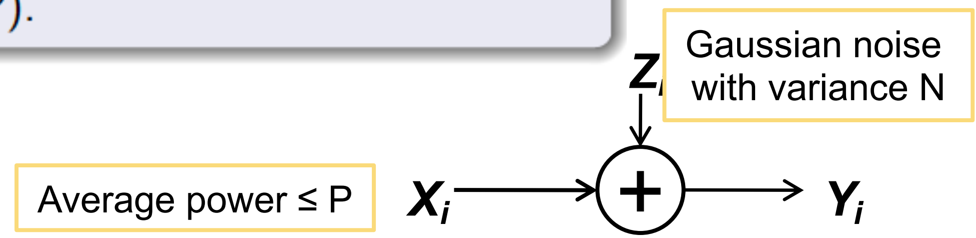
Capacity of time-discrete Gaussian channel

Capacity of Gaussian channel

The capacity of Gaussian channel with average input power $\leq P$ equals to:

$$C = \max_{E[X^2]\leq P} I(X;Y) = \frac{1}{2}\log(1+\frac{P}{N})$$

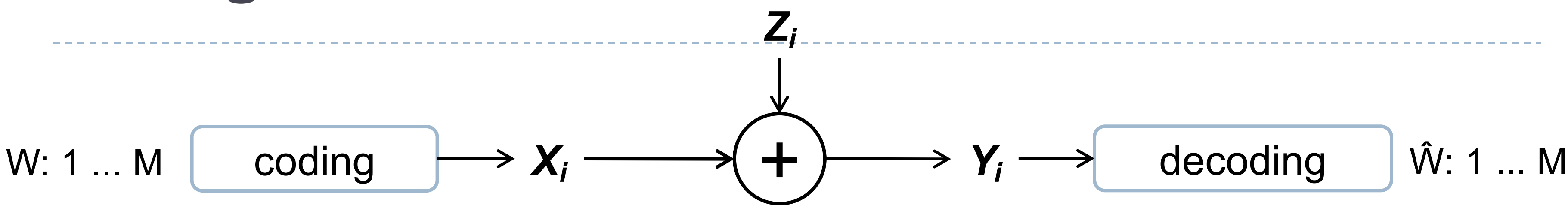
reaching the maximum value if $X \sim \mathcal{N}(0, P)$.



$$\begin{aligned} I(X;Y) &= h(Y) - h(Y|X) \\ &= h(Y) - h(X+Z|X) \\ &= h(Y) - h(Z|X) \\ &= h(Y) - h(Z), \end{aligned}$$

$$\begin{aligned} I(X;Y) &= h(Y) - h(Z) \\ &\leq \frac{1}{2}\log 2\pi e(P+N) - \frac{1}{2}\log 2\pi e N \\ &= \frac{1}{2}\log\left(1+\frac{P}{N}\right). \end{aligned}$$

Coding for Gaussian channel



Kodiranje za Gaussov kanal

Kodiranje (M, n) za Gaussov kanal z omejeno povprečno močjo vhodnega signala P je definirano na naslednji način:

1. Imamo zaporedje indeksov (simbolov) $\{1, 2, \ldots, M\}$.
2. Imamo funkcijo kodiranja $x : \{1, 2, \ldots, M\} \to \mathcal{X}^n$, s katero proizvedemo kode $x^n(1), x^n(2), \ldots, x^n(M)$, tako da zadoščajo pogoju omejitve moči:

$$\sum_{i=1}^{n} x_i^2(w) \leq nP, \quad w = 1, \ldots, M.$$

3. Imamo funkcijo dekodiranja

$$g : \mathcal{Y}^n \to \{1, 2, \ldots, M\}.$$

Probability of decoding error

Verjetnosti napake prenosa

1. *Pogojna verjetnost napake* (če je poslan indeks i, ki ni bil pravilno dekodiran):

$$\lambda_i = Pr(g(Y^n) \neq i | X^n = x^n(i)) = \sum_{y^n} p(y^n | x^n(i)) I(g(y^n) \neq i).$$

2. *Maksimalna verjetnost napake* za kodiranje (M, n) je

$$\lambda^{(n)} = \max_{i \in \{1, \ldots, M\}} \lambda_i .$$

3. *Povprečna verjetnost napake* za kodiranje (M, n) je

$$P_e^{(n)} = \frac{1}{M} \sum_{i=1}^{M} \lambda_i .$$

Coding rate

Coding rate

The coding rate R of (M, n) encoding equals to:

$$R = \frac{\log M}{n}$$

Shannon result on coding of Gaussian channel

Coding rate for Gaussian channel - Shannon

The upper bound of coding rate R for (M, n) encoding, over a Gaussian channel with variance N and average signal power P, **cannot exceed** the channel capacity:

$$C = \frac{1}{2}\log(1 + \frac{P}{N}) \quad \text{bits per channel use}$$

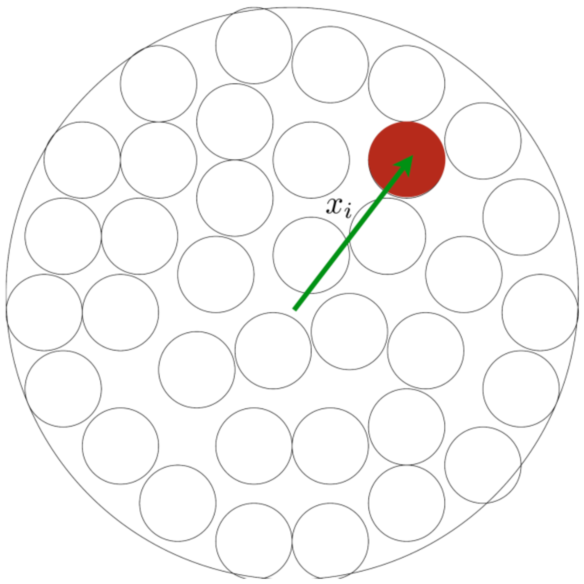
In brief: $R < C$ (if we want error-free communication).

Gaussian normal distributed noise with mean value 0 is the worst possible case !!!!

Shannon result on coding of Gaussian channel: proof (optional –similar to other cases)

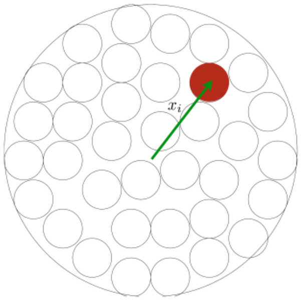
- ***Skica dokaza:***

- Vzemimo, da želimo konstruirati $(2^{nR}, n)$ kode z zelo majhno napako dekodiranja po prenosu.
- Kaj se dogaja z vhodnim nizom x_i dolžine n pri prenosu. Sprejeti vektor (tudi dolžine n) je normalno porazdeljen s povprečjem enakim vhodnemu nizu in varianco enako varianci šuma.
- To pomeni, da z veliko verjetnostjo lahko trdimo, da bo sprejeti niz (vektor), vsebovan v krogli z radijem $\sqrt{n(N+\epsilon)}$ in središčem v točki x_i.
- Če izvedemo dekodiranje tako, da vse nize, ki ležijo v taki krogli, pripišemo vhodnemu nizu x_i, potem naredimo napako samo v primeru, ko se vhodni niz ne prenese v našo kroglo. Ta verjetnost pa je zelo majhna.



Shannonov izrek o kodiranju Gaussovega kanala: dokaz (optional)

- ***Skica dokaza:***



Preštejmo, koliko takšnih krogel je mogoče postaviti v Gaussovem kanalu z omejeno močjo vhodnega signala P.

- Volumen n-dimenzionalne krogle je oblike $C_n r^n$, kjer je r radij krogle.
- V našem primeru imamo krogle $\sqrt{n(N+\epsilon)} \approx \sqrt{nN}$.
- Sprejeti vektorji imajo moč, ki ni večja od $n(P+N)$, torej ležijo v krogli z radijem $\sqrt{n(P+N)}$.
- Koliko krogel z radijem $\sqrt{nN}$ lahko postavimo v kroglo z radijem $\sqrt{n(P+N)}$? Toliko simbolov lahko namreč z minimalno napako prenesemo naenkrat po kanalu.

$$\frac{C_n(n(P+N))^{\frac{n}{2}}}{C_n(nN)^{\frac{n}{2}}} = 2^{\frac{n}{2}\log\left(1+\frac{P}{N}\right)}$$

- Pri kodi $(2^{nR}, n)$ prenašamo naenkrat po kanalu 2^{nR} simbolov. Torej mora biti $2^{nR} \leq 2^{\frac{n}{2}\log\left(1+\frac{P}{N}\right)}$, ali $R \leq \frac{1}{2}\log\left(1+\frac{P}{N}\right) = C$, C je kapaciteta Gaussovega kanala.

Watter filling - introduction

- Imagine the problem of bying gift items for Christmas !

Item	Unit Price	Quantity
	$2	?
	$10	?
	$5	?
	$329	?

Budget:
$500

Budget allocation problem

- total budget: w, money allocated for nth item: w_n, unit price for nth item: p_n, can buy w_n/p_n items

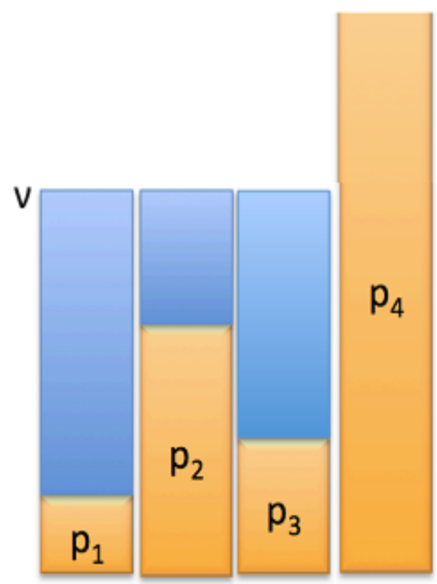
- Goal: buy as many gift as possible, but also want to diversify. Diminishing return on the number of items bought $\log(1 + w_n/p_n)$

- budget allocation problem

$$\begin{aligned} \max_{w_n} \quad & \sum_{n=1}^{N} \log(1 + w_n/p_n) \\ \text{subject to} \quad & \sum_{n=1}^{N} w_n = W \\ & w_n \geq 0 \end{aligned}$$

Optimal solution – watter filling

- $w_n = (\nu - p_n)^+$, $(x)^+ = x$ if $x \geq 0$, 0 otherwise
- ν determined by budget constraint: $\sum_{n=1}^{N} (\nu - p_n)^+ = w$



Application to parallel Gaussian channels

- Channel capacity of parallel Gaussian channel can be formulated into a similar problem

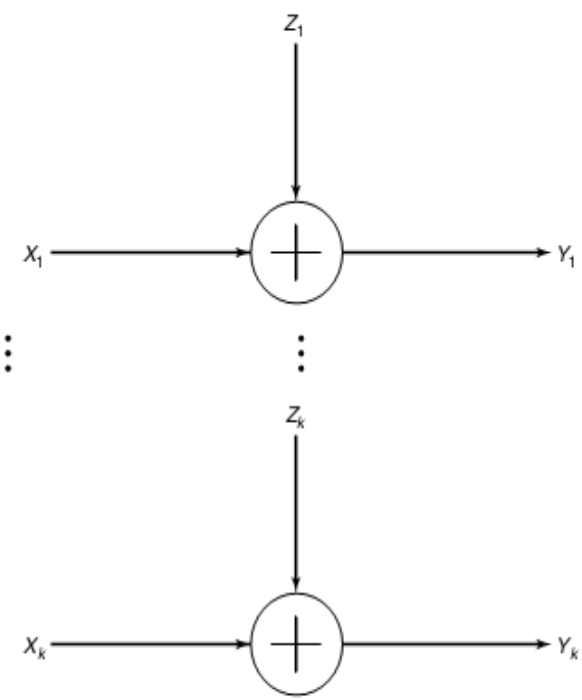
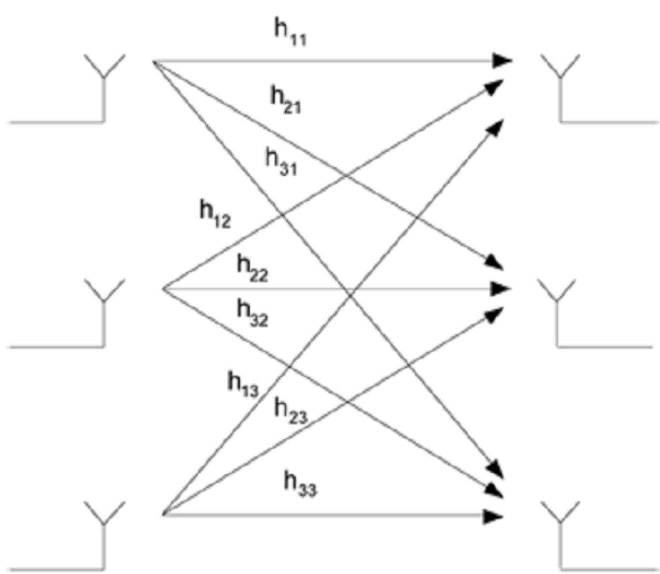


FIGURE 9.3. Parallel Gaussian channels.

Where do we find parallel channels ?

everywhere:

- OFDM (orthogonal frequency-division multiplexing), parallel channels formed in frequency domain
- MIMO (multiple-input-multiple-output) – multiple antenna system
- DSL (or discrete multi-tone systems)



Parallel independent Gaussian channels

- k **independent** channels
- $Y_i = X_i + Z_i,\ i = 1, 2, \ldots, k,\ Z_i \sim \mathcal{N}(0, N_i)$
- total power constraint $E \sum_{i=1}^{k} X_i^2 \leq P$
- goal: distribute power among various channels to maximize the total capacity

Channel capacity

- channel capacity of parallel Gaussian channel

$$C = \max_{f(x_1,\ldots,x_k):\sum EX_i^2 \le P} I(X_1,\ldots,X_k;Y_1,\ldots,Y_k)$$

$$C = \frac{1}{2}\sum_{i=1}^{k} \log(1 + \frac{P_i}{N_i})$$

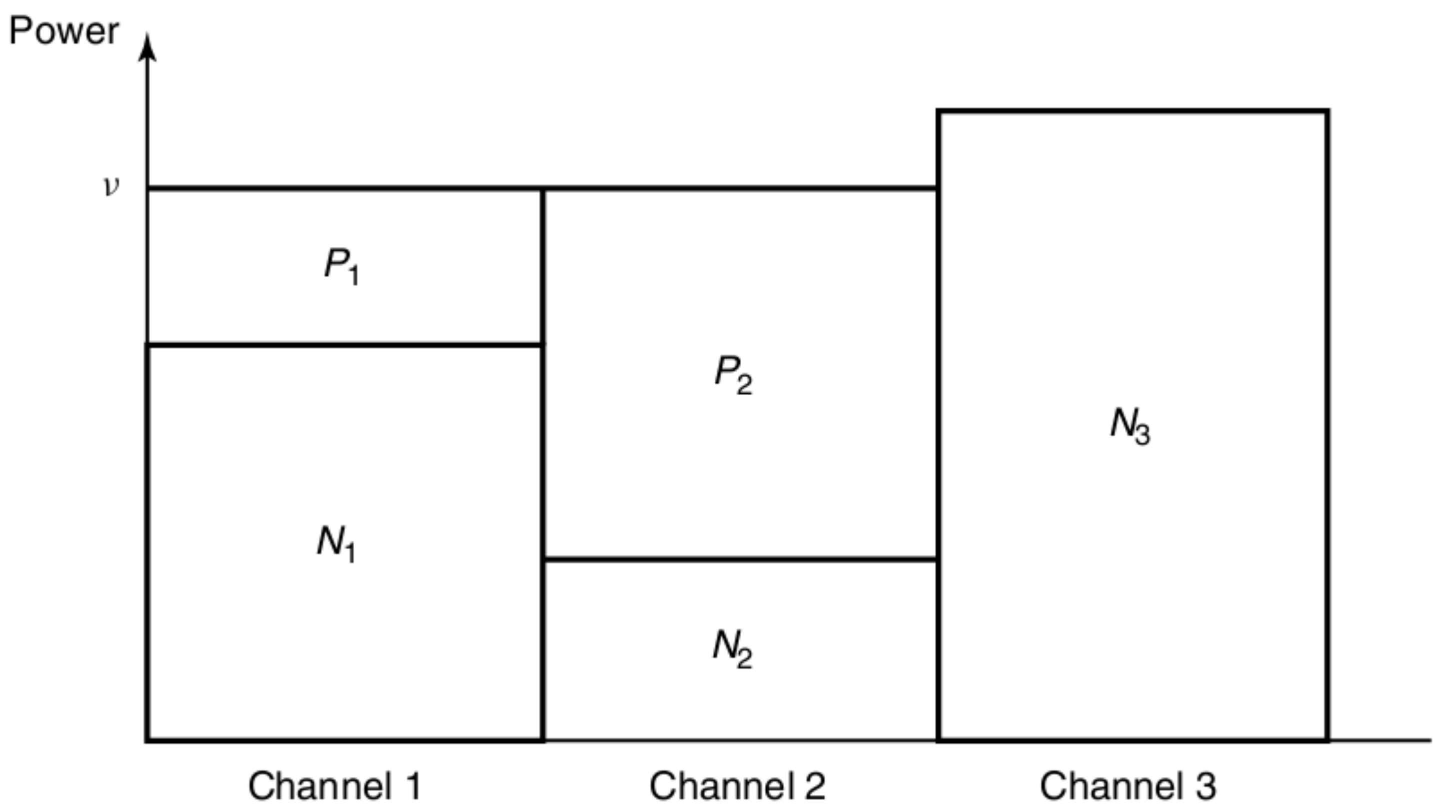
- power allocation problem

$$\max_{P_i} \quad \sum_{i=1}^{k} \log(1 + P_i/N_i)$$

$$\text{subject to} \quad \sum_{i=1}^{k} P_i = P$$

$$P_i \ge 0$$

Water-filling solution

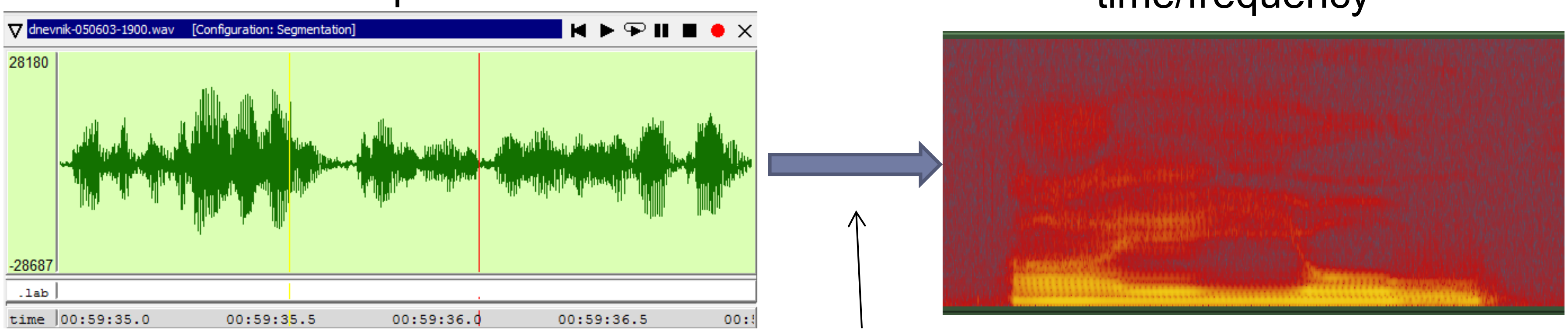


Frequency limited Gaussian channels

- Frequency limited signal is a continous time function (not limited), with limits in frequency.
- Example: speach signal (commonly placed between 300Hz and 4-5KHZ)

time/amplitude

time/frequency



Fourier transformation

Frequency limited Gaussian channels

- Example: radio connections, telephone lines

Model of frequency limited Gaussian channel

This channel is modelled as:

$$Y(t) = (X(t) + Z(t)) * h(t)$$

where all signals are **time continuous** and $X(t)$ is the input to a channel with Gaussian noise $Z(t)$ and $h(t)$ is the time response of a bandpass filter which cuts frequencies above W. This filtering is described as **convolution in time** or alternatively **product in frequency domain**.

Frequency limited signals

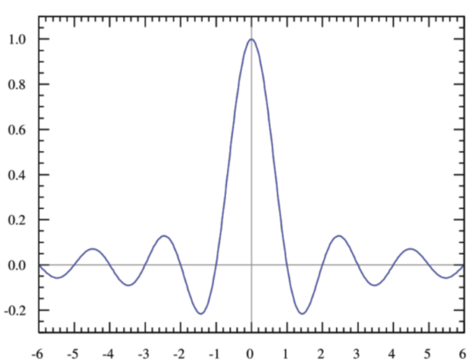
Filtriranje signala s frekvenčno omejenim filtrom: →Filtering of signal using frequency range W

$$y(t) = x(t) * h(t).$$

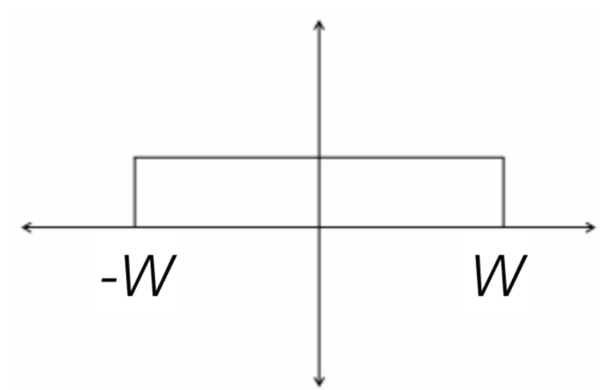
Fourier transformation

$$Y(\omega) = X(\omega) \cdot H(\omega).$$

h(t)



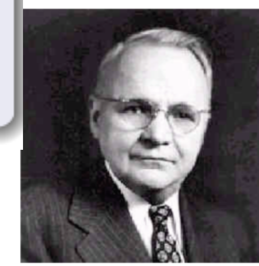
H(ω)

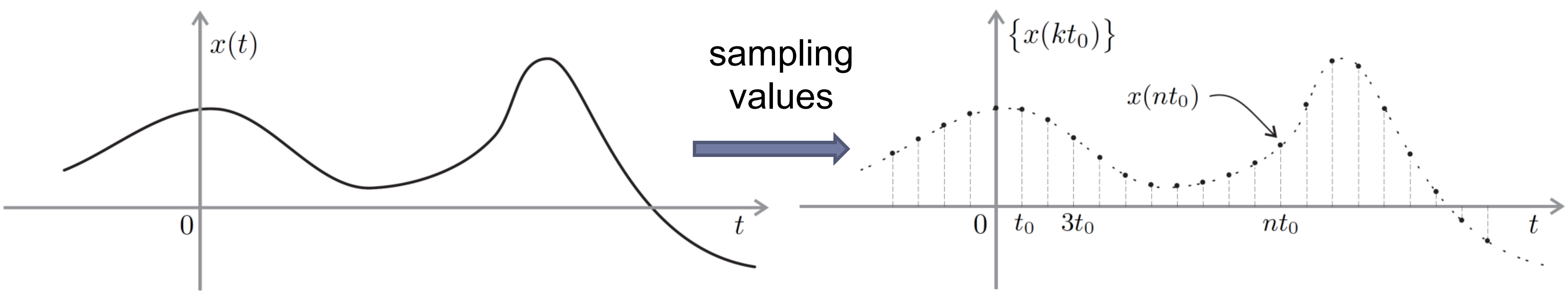


Sampling of frequency limited signals - Nyquist

Nyquist result on sampling

- We assume that we have a time continuous signal $x(t)$ limited in frequency by W, thus having frequencies from 0 to W Hz.
- We can make a discrete version of the signal by using $2W$ samples per second (sampling frequency is $2W$)! Nyquist says: We can reconstruct such signal “without distortion” !?





Capacity of frequency limited Gaussian channels

- We assume a Gaussian channel with frequency limit W. This means we can transmit signals from 0 to W hertz.

- Assume that the **input signal has duration** T in seconds:
– Nyquist: Sampling the input signal $2WT$ samples (can recover later)
– Input signal becomes a vector of dimension $n = 2WT$
–The power of input signal is limited to P. It means that the energy of signal is limited by PT. The energy per sample is then $PT/n = \frac{1}{2}P/W$.

- Now we treat the **noise signal**: –Noise is also limited by W in frequency
– Its energy is defined as $N_0/2$ Watt/Hertz. The power of noise is $N_0/22W = N_0W$. The (average) energy of noise per sample for a signal of duration T is: $N_0WT/n = N_0/2$.

Capacity of frequency limited Gaussian channels

- Izračunali smo že kapaciteto diskretnega Gaussovega kanala

$$C = \frac{1}{2}\log\left(1 + \frac{P}{N}\right).$$ discrete Gaussian channel

- Torej je kapaciteta enega odtipka frekvenčno omejenega zveznega Gaussovega kanala: (capacity per sample of frequency limited Gaussian channel)

$$C = \frac{1}{2}\log\left(1 + \frac{P/2W}{N_0/2}\right) = \frac{1}{2}\log\left(1 + \frac{P}{N_0 W}\right)$$ bit per sample bitov na odtipek.

- Ker imamo $2W$ odtipkov na sekundo, je tako **kapaciteta frekvenčno omejenega zveznega Gaussovega kanala:**

$$C = W\log\left(1 + \frac{P}{N_0 W}\right)$$ bitov na sekundo.

capacity of frequency limited Gaussian channel

Capacity of frequency non-limited Gaussian channel

- Capacity of frequency limited Gaussian channel :

$$C = W \log\left(1 + \frac{P}{N_0 W}\right) \quad \text{bitov na sekundo.}$$

- What happens in case $W \to \infty$

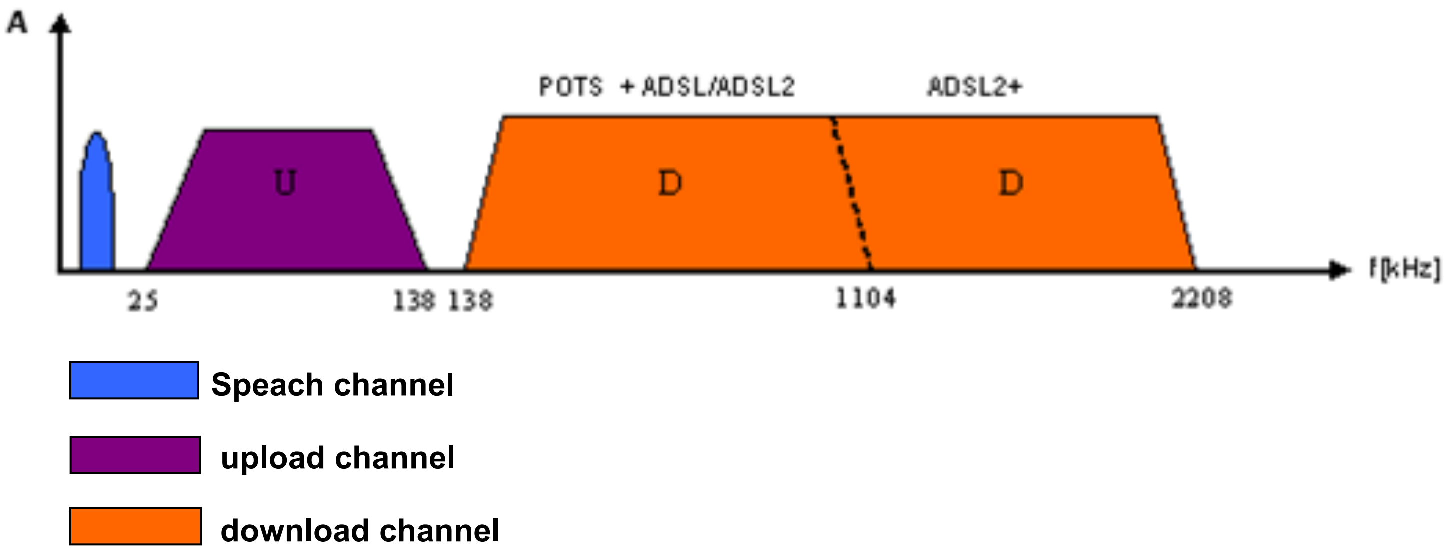
$$W_0 = P/N_0 \Rightarrow \frac{C}{W_0} = \frac{W}{W_0}\log(1+\frac{W_0}{W}) \to_{W\to\infty} = \log e$$

$$\boxed{C = \frac{P}{N_0}\log_2 e \quad \text{bitov na sekundo}}$$

- Thus:
  - **Capacity of frequency non-limited channel grows linearly with signal power.**

Example: wired connection and data transmission

- Frequency range for xDSL technology



povzeto po B. Batagelj, ULJ, FE

Example: wired connection and data transmission

Telephone modem uses the frequency range

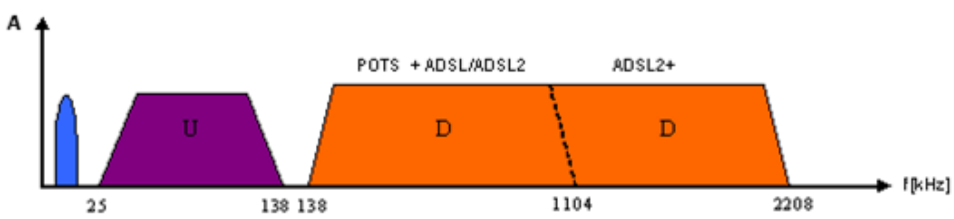
$f = 300\,\text{Hz}....3400\,\text{Hz}$ ⇒ W=3,1 kHz

and requires signal to noise ratio (SNR) 35 dB.

$$10\log_{10}\frac{P}{N_0W} = 35\,\text{dB} \Rightarrow \frac{P}{N_0W} = 10^{\frac{35}{10}} = 3162$$

$$C = W\cdot\log_2\left(1+\frac{P}{N_0W}\right) = 3{,}1\,\text{kHz}\cdot\log_2(1+3162)$$

$$C = 3{,}1\cdot10^3\,\text{s}^{-1}\cdot11{,}6\,\text{bit} = 36\,\text{kbit/s}$$



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Example: ADSL upload

ADSL modem has upload in frequency range

$$f = 25\,\text{kHz}....138\,\text{kHz} \Rightarrow W = 113\text{ kHz}$$

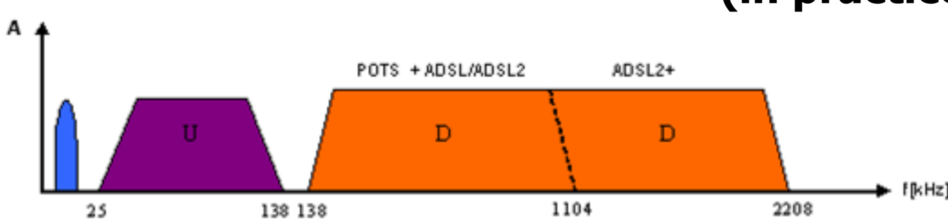
and requires SNR 35 dB.

$$10\log_{10}\frac{P}{N_0W} = 35\,\text{dB} \Rightarrow \frac{P}{N_0W} = 10^{\frac{35}{10}} = 3162$$

$$C = W \cdot \log_2\left(1 + \frac{P}{N_0W}\right) = 113\,\text{kHz} \cdot \log_2(1 + 3162)$$

$$C = 113 \cdot 10^3 \text{s}^{-1} \cdot 11{,}6\,\text{bit} = 1{,}3\,\text{Mbit/s}$$

theoretical
(in practice – almost halved)



povzeto po B. Batagelj, ULJ, FE

Example: ADSL download

ADSL modem has download in frequency range

$$f = 138\text{kHz}\ldots1104\text{kHz} \;\Longrightarrow\; W\text{=966 kHz}$$

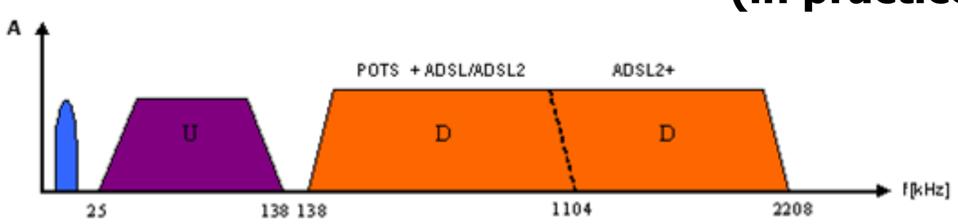
and the request is that SNR is 35 dB.

$$10\log_{10}\frac{P}{N_0W} = 35\text{ dB} \Rightarrow \frac{P}{N_0W} = 10^{\frac{35}{10}} = 3162$$

$$C = W\cdot\log_2\left(1+\frac{P}{N_0W}\right) = 966\text{ kHz}\cdot\log_2(1+3162)$$

$$C = 996\cdot10^3\text{s}^{-1}\cdot11{,}6\text{ bit} = 11{,}2\text{ Mbit/s}$$

theoretical
(in practice about half)



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Spectral efficiency

$$C/W = \log_2\left(1 + \frac{P}{N_0 W}\right) \quad \text{[bit/s/Hz = bit]}$$

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year	Channel capacity C	Freq. band W	Spectral efficiency C/W
1900 telegraphy (ear reception)	10 bit/s	500 Hz	0,02 bit/s/Hz
1950 Radio teleprinter	50 bit/s	250 Hz	0,2 bit/s/Hz
1990 GSM telephony	271 kbit/s	200 kHz	1,355 bit/s/Hz
1960 analog TV	100 Mbit/s	7 MHz	14 bit/s/Hz
today			0,3 ..... 10 bit/s/Hz
Satelite connections			1bit/s/Hz