

Solved Problems - TOR III - 22.05.2020

1. Let C be the $(5, 3)$ binary code with generator matrix

$$G = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Find a standard form generator matrix for a code equivalent to C .

► **Solution.** Do the following sequence of elementary row operations on G : (1) interchange Row 1 and Row 2, (2) interchange Row 2 and Row 3; (3) add Row 2 to Row 1, (4) add Row 2 to Row 3. At the end of this chain of row operations, the matrix

$$G' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

is obtained. This matrix is a standard form generator matrix of the same code C . ◀

2. Let C be the ternary linear code with generator matrix

$$G = \begin{bmatrix} 1 & 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

List all the codewords in C and find the weight of each codeword. What is $d(C)$?

► **Solution.** All of the codewords in C are given by the matrix multiplications $\hat{y}G \in GF(3)^5$ where $\hat{y} = y_1y_2 \in GF(3)^2$. There are 9 such words. The words and their weights are recorded in the following table.

\hat{y}	$\hat{x} = \hat{y}G$	$w(\hat{x})$
00	00000	0
01	01101	3
02	02202	3
10	10220	3
11	11021	4
12	12122	5
20	20110	3
21	21211	5
22	22012	4

The minimum weight of a codeword is 3 so $d(c) = 3$. ◀

3. Construct a (6, 3, 3) binary code.

► **Solution.** We will construct such a code by producing a parity check matrix H . A parity check matrix of a (6, 3) code is a 3×6 binary matrix of rank 3. In standard form, the last three columns will form the 3×3 identity matrix. According to Theorem the code will have $d(C) = 3$ provided all of the columns of H are nonzero and distinct and some set of 3 columns adds to 0. Thus, all we need to do is add 3 different columns to the identity matrix I_3 . For example,

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

is such a matrix. Notice that all columns are different and columns 1, 4, and 5 add up to 0. Thus, $d(C) = 3$ for the linear code C with this parity check matrix. Since this matrix $H = [A \ I_3]$, it follows that a generator matrix of C is $G = [I_3 \ -A^T]$. Thus, a generator matrix for C is

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$



4. Construct a standard array for the (5,2) binary array with generator matrix

$$G = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Use your array to decode the received strings $\hat{y}_1 = 11101$ and $\hat{y}_2 = 01001$.

► **Solution.** The standard array is:

Coset Leader				
00000	10110	01011	11101	
00001	10111	01010	11100	
00010	10100	01001	11111	
00100	10010	01111	11001	
01000	11110	00011	10101	
10000	00110	11011	01101	
10001	00111	11010	01100	
11000	01110	10011	00101	

Using this array, $\hat{y}_1 = 11101$ is in the first row, and hence is already a code word, while $\hat{y}_2 = 01001$ is in the third column and hence is decoded as the codeword 01011 that is at the top of the third column.



5. Let C be the (4,2) binary code with parity check matrix

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

a) Is this a 1-error-correcting code? Is this a 1-error-detecting code?

► **Solution.** minimum distance of this code is $d = 1$ since the second column of H is $\hat{0}$ which is linearly dependent. Thus, C can detect $0 = d - 1$ errors and it can correct $0 = (d - 1)/2$ errors. Hence it is neither 1-error-correcting and 1-error-detecting. ◀

b) Construct a syndrome table for C .

► **Solution.** The syndrome table is:

Coset Leader	Syndrome
0000	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
0001	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
0010	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
0011	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
1000	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

c) Use your table to decode the received strings $\hat{y}_1 = 1011$ and $\hat{y}_2 = 0011$.

► **Solution.** $H\hat{y}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so \hat{y}_1 is a codeword. $H\hat{y}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which is in the fourth row of the syndrome table. Thus, it is decoded by changing the first digit of the word, so the decoded string is 1011. ◀

6. Let C be the (4,2) binary code with parity check matrix

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Construct a syndrome table for C and use your table to decode the received strings $\hat{y}_1 = 0100$ and $\hat{y}_2 = 1011$.

► **Solution.** The syndrome table is:

Coset Leader	Syndrome
0000	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
0001	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
0010	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
0100	$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$H\hat{y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ which is in the fourth row of the table, and so it decodes by changing the second digit to get 0000 as the decoded string. $H\hat{y}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ which is in the second row of the table, and so it decodes by changing the last digit to get 1010 as the decoded string. ◀

7. Let C_1 and C_2 be binary linear codes having the generator matrices

$$G_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad G_2 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

(a) List the codewords of C_1 and C_2

► **Solution.** All of the code words are $\hat{y}G_1$ for C_1 and $\hat{y}G_2$ for C_2 . Thus, the codewords and their weights are:

C_1			C_2		
\hat{y}	$\hat{x} = \hat{y}G_1$	$w(\hat{x})$	\hat{y}	$\hat{x} = \hat{y}G_2$	$w(\hat{x})$
00	00000	0	000	0000000	0
01	00111	3	001	0010111	4
10	11110	4	010	0101011	4
11	11001	3	011	0111100	4
			100	1001101	4
			101	1011010	4
			110	1100110	4
			111	1110001	4

(b) Determine the minimum distance of each code. ◀

► **Solution.** Since $d(c)$ is the minimum weight of a nonzero codeword, from the above tables we get $d(C_1) = 3$ and $d(C_2) = 4$. ◀

8. Let C be the binary linear code with generator matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

(a) Find a generator matrix G for C in standard form.

► **Solution.** Row reduce the given matrix to get the generator matrix in standard form:

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

(b) Find a parity check matrix H for C .

► **Solution.** From the standard form of $G = [I_4 \ A]$ we can compute the parity check matrix as $H = [-A^T \ I_3]$. Hence

$$H = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

(c) Determine the minimum distance $d(C)$ of C .

► **Solution.** Since all the columns of H are distinct, but columns 1, 6, and 7 add to 0, it follows that $d(C) = 3$.

(d) How many errors can C detect? How many can it correct?

► **Solution.** Since $d(C) = 3$, C can detect 2 errors and correct 1 error.

9. Recall that the 10 digit ISBN code is a 10 digit string of words in $GF(11) = \mathbb{Z}_{11}$

$$x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}$$

with $\sum_{i=1}^{10} ix_i \equiv 0 \pmod{11}$. Replace ? with the correct digit in the following ISBN: 368576?969.

► **Solution.** The seventh digit is missing, so

$$7x_7 \equiv -(3 + 2 \cdot 6 + 3 \cdot 8 + 4 \cdot 5 + 5 \cdot 7 + 6 \cdot 6 + 8 \cdot 9 + 9 \cdot 6 + 10 \cdot 9) \pmod{11}.$$

Thus, $7x_7 \equiv 6 \pmod{11}$. Since the inverse of 7 mod 11 is 8 it follows that $x_7 \equiv 8 \cdot 6 \equiv 4 \pmod{11}$. Thus, the missing digit is 4.

10. Assume that the code Hamming[15,11] is being used. Decode the following received strings.

(a) 110111011110111

► **Solution.** For this string \hat{y} , $H\hat{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, which is the eighth column of H .

Thus, \hat{y} is decoded by changing the eighth binary digit of \hat{y} . Thus, the decoded string is 110111001110111. ◀

(b) 001000100001000

► **Solution.** For this string \hat{y} , $H\hat{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, which is the eighth column of H .

Thus, \hat{y} is decoded by changing the eighth binary digit of \hat{y} . Thus, the decoded string is 001000110001000. ◀

11. Let C be the ternary linear code having the generator matrix

$$G = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

(a) Find the standard form parity check matrix for C .

► **Solution.**

$$H = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix}.$$

(b) Write a syndrome table for the code C . ◀

► **Solution.**

Coset Leader	Syndrome
0000	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
0001	$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$
0002	$\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$
0010	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
0020	$\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$
0100	$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$
0200	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
1000	$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$
2000	$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$



(c) Use syndrome decoding to decode the received vectors 2121, 1201, and 2222.

► **Solution.** $H\hat{y}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ which is in the second row of the table. Thus, the string 2121 is decoded by subtracting 1 from the fourth ternary digit. Hence, 2121 decodes to 2120.

$H\hat{y}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ which is in the last row of the table. Thus, the string 1201 is decoded by subtracting 2 from the first ternary digit. Hence, 1201 decodes to 2201.

$H\hat{y}_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ which is in the last row of the table. Thus, the string 2222 is decoded by subtracting 2 from the first ternary digit. Hence, 2222 decodes to 0222. ◀

12. Consider the [6,3] code with the generator matrix:

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Find all the cosets, their syndromes and minimal weights in each coset.

Solution: First we need to apply the row reduction to G (subtract the 3rd row from the 2nd):

$$G' = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

The parity check matrix then equals

$$H = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We can use it to compute the syndromes of all vectors and list all the cosets:

Leader	Coset	Syndrome	Min. wt
000000	000000,100011,010111,001110,110100,101101,011001,111010	000	0
100000	100000,000011,110111,101110,010100,001101,111001,011010	011	1
010000	010000,110011,000111,011110,100100,111101,001001,101010	111	1
001000	001000,101011,011111,000110,111100,100101,010001,110010	110	1
000100	000100,100111,010011,001010,110000,101001,011101,111110	100	1
000010	000010,100001,010101,001100,110110,101111,011011,111000	010	1
000001	000001,100010,010110,001111,110101,101100,011000,111011	001	1
101000	101000,001011,111111,100110,011100,000101,110001,010010	101	2

13. Is there a [12,7,5] binary code (that is, $n = 12$, $k = 7$ and the minimal distance between codewords is 5)?

Solution: Suppose that such code exists. Since the minimal distance between codewords is $d = 5$, then this code corrects $t = (5 - 1)/2 = 2$ errors, and we can apply the Sphere-Packing Bound:

$$\left(1 + 12 + \binom{12}{2}\right) 2^7 \leq 2^{12}, 1 + 12 + 66 = 79 \leq 2^5 = 32.$$

Contradiction, so there is no such code.

14. Prove that any binary $[23,12,7]$ code is perfect. How many errors can it correct?

Solution: Since the minimal distance between codewords is $d = 7$, then this code corrects $t = (7 - 1)/2 = 3$ errors. We can apply the Sphere-Packing Bound:

$$\left(1 + 23 + \binom{23}{2} + \binom{23}{3}\right) 2^{12} \leq 2^{23}.$$

Remark that

$$1 + 23 + \binom{23}{2} + \binom{23}{3} = 1 + 23 + \frac{23 \cdot 22}{2} + \frac{23 \cdot 22 \cdot 21}{6} = 1 + 23 + 253 + 1771 = 2048,$$

and $2048 \cdot 2^{12} = 2^{11} \cdot 2^{12} = 2^{23}$. Since we get an equality in the Sphere-Packing Bound, the code is perfect.

15. Decode the message $y = 1001001$, which was encoded using the $[7,4]$ Hamming code with the generator matrix

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

and at most one bit was changed.

Solution: The parity check matrix has the form:

$$H = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Let us check all possible options for x (different from y in at most one bit) and compute their syndromes:

x	1001001	0001001	1101001	1011001	1000001	1001101	1001011	1001000
$syn(x)$	101	110	000	011	010	001	111	100

A word x is a codeword if and only if its syndrome equals to 0, so the corresponding codeword is 1101001. The original message consists of the first 4 bits of x and reads as 1101.