Linear Block Code Analysis and Decoding Solution

Problem: Let the following parity-check equations be used to specify a binary linear code of length 7. Thus, its codewords $(c_1, \ldots, c_7)$ satisfy

$$c_1 + c_2 + c_5 = 0,$$

$$c_2 + c_3 + c_6 = 0,$$

$$c_1 + c_2 + c_3 + c_7 = 0,$$

$$c_4 + c_5 + c_6 + c_7 = 0.$$

(a) How many codewords does this code have, and what is its minimum distance? How many errors can this code detect? How many errors can it correct?

(b) Write a generator matrix for this code.

(c) Construct a syndrome table for this code. Use your table to decode the received strings

$$y_1 = (0, 0, 1, 1, 1, 0, 1)$$

and

$$y_2 = (0, 0, 0, 0, 0, 0, 1).$$

Solution

a: Let the given parity-check equations for a binary linear code of length $n = 7$ be:

$$\begin{aligned} c_1 \oplus c_2 \oplus c_5 &= 0 \\ c_2 \oplus c_3 \oplus c_6 &= 0 \\ c_1 \oplus c_2 \oplus c_3 \oplus c_7 &= 0 \\ c_4 \oplus c_5 \oplus c_6 \oplus c_7 &= 0 \end{aligned}$$

We can express these equations in matrix form $H\mathbf{c}^T = \mathbf{0}$, where $\mathbf{c} = (c_1, c_2, c_3, c_4, c_5, c_6, c_7)$. The parity-check matrix H is:

$$H = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

To identify the standard systematic properties, we bring H into Row-Reduced Echelon Form (RREF) over GF(2):

- Add Row 1 to Row 3: $\text{Row}_3 \leftarrow \text{Row}_3 \oplus \text{Row}_1 = (0, 0, 1, 0, 1, 0, 1)$

- Add Row 3 to Row 2: $\text{Row}_2 \leftarrow \text{Row}_2 \oplus \text{Row}_3 = (0,1,0,0,1,1,1)$
- Add Row 2 to Row 1: $\text{Row}_1 \leftarrow \text{Row}_1 \oplus \text{Row}_2 = (1,0,0,0,0,1,1)$

This yields the systematic form $H_{\text{RREF}} = [I_4 \mid P]$:

$$H_{\text{RREF}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The rank of the matrix is $r = 4$. Thus, the dimension of the code is $k = n - r = 7 - 4 = 3$.

(a) Code Parameters, Error Detection, and Correction

- **Number of Codewords:** The code contains $2^k = 2^3 = 8$ codewords.
- **Minimum Distance (d_{min}):** The minimum distance equals the minimum number of linearly dependent columns in H. No individual column is zero, and no two columns are identical. Testing combinations shows columns 1, 3, and 7 sum to zero over GF(2). Hence, $d_{\text{min}} = 3$.
- **Error Detection Capability (t_{detect}):**

$$t_{\text{detect}} = d_{\text{min}} - 1 = 3 - 1 = 2 \text{ errors}$$

- **Error Correction Capability (t_{correct}):**

$$t_{\text{correct}} = \left\lfloor \frac{d_{\text{min}} - 1}{2} \right\rfloor = \left\lfloor \frac{3 - 1}{2} \right\rfloor = 1 \text{ error}$$

(b) Generator Matrix

Since $H_{\text{RREF}} = [I_4 \mid P]$, the systematic generator matrix G is constructed as $G = [P^T \mid I_3]$:

$$P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \implies G = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

(c) Syndrome Table and Decoding

The syndrome is defined as $\mathbf{S} = H\mathbf{y}^T$. Below is the standard syndrome table mapped to the lowest-weight coset leaders (error patterns):

Table 1: Syndrome Table sorted by weight

Error Pattern e	**Syndrome** $\mathbf{S}^T = (s_1, s_2, s_3, s_4)$	**Weight**
$(0,0,0,0,0,0,0)$	$(0,0,0,0)$	0
$(0,0,0,1,0,0,0)$	$(0,0,0,1)$	1
$(0,0,0,1,0,0,1)$	$(0,0,1,0)$	2
$(0,0,0,0,0,0,1)$	$(0,0,1,1)$	1
$(1,1,0,0,0,0,0)$	$(0,1,0,0)$	2
$(0,0,0,0,0,1,0)$	$(0,1,0,1)$	1
$(0,0,1,0,0,0,0)$	$(0,1,1,0)$	1
$(0,1,0,0,1,0,0)$	$(0,1,1,1)$	2
$(0,1,1,0,0,0,0)$	$(1,0,0,0)$	2
$(0,0,0,0,1,0,0)$	$(1,0,0,1)$	1
$(1,0,0,0,0,0,0)$	$(1,0,1,0)$	1
$(1,0,0,1,0,0,0)$	$(1,0,1,1)$	2
$(1,0,1,0,0,0,0)$	$(1,1,0,0)$	2
$(0,1,0,0,0,0,1)$	$(1,1,0,1)$	2
$(0,1,0,0,0,0,0)$	$(1,1,1,0)$	1
$(1,0,0,0,0,1,0)$	$(1,1,1,1)$	2

Decoding $\mathbf{y_1} = (0,0,1,1,1,0,1)$

Compute the syndrome vector $\mathbf{S_1} = H\mathbf{y_1}^T$:

$$\begin{aligned} s_1 &= 0 \oplus 0 \oplus 1 = 1 \\ s_2 &= 0 \oplus 1 \oplus 0 = 1 \\ s_3 &= 0 \oplus 0 \oplus 1 \oplus 1 = 0 \\ s_4 &= 1 \oplus 1 \oplus 0 \oplus 1 = 1 \end{aligned}$$

Thus, $\mathbf{S_1} = (1,1,0,1)^T$. From the syndrome table, this matches the error pattern $\mathbf{e} = (0,1,0,0,0,0,1)$.
The decoded codeword is:

$$\mathbf{c_1} = \mathbf{y_1} \oplus \mathbf{e} = (0,0,1,1,1,0,1) \oplus (0,1,0,0,0,0,1) = (0,1,1,1,1,0,0)$$

Decoding $\mathbf{y_2} = (0,0,0,0,0,0,1)$

Compute the syndrome vector $\mathbf{S_2} = H\mathbf{y_2}^T$:

$$\begin{aligned} s_1 &= 0 \oplus 0 \oplus 0 = 0 \\ s_2 &= 0 \oplus 0 \oplus 0 = 0 \\ s_3 &= 0 \oplus 0 \oplus 0 \oplus 1 = 1 \\ s_4 &= 0 \oplus 0 \oplus 0 \oplus 1 = 1 \end{aligned}$$

Thus, $\mathbf{S_2} = (0,0,1,1)^T$. From the syndrome table, this matches the error pattern $\mathbf{e} = (0,0,0,0,0,0,1)$.
The decoded codeword is:

$$\mathbf{c_2} = \mathbf{y_2} \oplus \mathbf{e} = (0,0,0,0,0,0,1) \oplus (0,0,0,0,0,0,1) = (0,0,0,0,0,0,0)$$