Exercise: Water-Filling for Three Parallel Channels

Problem Statement

We consider $K = 3$ parallel independent Gaussian channels (AWGN) with different interference/noise powers σ_k^2, where:

- Channel 1: $\sigma_1^2 = 2$
- Channel 2: $\sigma_2^2 = 4$
- Channel 3: $\sigma_3^2 = 7$

The total transmission power allocation is constrained by $\sum_{k=1}^{3} P_k \leq P_X$. The total mutual information (system capacity) in "bits" is given by:

$$C = \frac{1}{2} \sum_{k=1}^{3} \log_2 \left(1 + \frac{P_k}{\sigma_k^2}\right) \tag{1}$$

Questions

1. **Scenario A:** Find the optimal power allocation (P_1, P_2, P_3) and total channel capacity C when the total power budget is $P_X = 11$.
2. **Scenario B:** Find the optimal power allocation (P_1, P_2, P_3) and total channel capacity C when the total power budget is reduced to $P_X = 3$.

Solution

According to the water-filling principle, power is allocated such that the sum of the noise baseline and the allocated transmission power is constant across all active channels:

$$P_k = \max(0, H - \sigma_k^2) \tag{2}$$

where H represents the uniform "water level".

1. Solution for Scenario A ($P_X = 11$)

First, we assume all three channels receive a positive power allocation ($P_k > 0$ for all k).

1. **Set up the power constraint equation:**

$$(H - \sigma_1^2) + (H - \sigma_2^2) + (H - \sigma_3^2) = P_X$$
$$(H - 2) + (H - 4) + (H - 7) = 11$$

2. **Solve for the water level H:**

$$3H - 13 = 11$$
$$3H = 24 \implies \mathbf{H = 8}$$

3. **Calculate individual power allocations:**

- $P_1 = 8 - 2 = \mathbf{6}$
- $P_2 = 8 - 4 = \mathbf{4}$
- $P_3 = 8 - 7 = \mathbf{1}$

Since all calculated powers are positive ($P_k > 0$), our initial assumption is valid.

4. **Calculate total system capacity C:**

$$C = \frac{1}{2}\left[\log_2\left(1 + \frac{6}{2}\right) + \log_2\left(1 + \frac{4}{4}\right) + \log_2\left(1 + \frac{1}{7}\right)\right]$$
$$C = \frac{1}{2}\left[\log_2(4) + \log_2(2) + \log_2\left(\frac{8}{7}\right)\right]$$
$$C \approx \frac{1}{2}\left[2 + 1 + 0.1926\right] = \mathbf{1.5963} \text{ bits}$$

2. Solution for Scenario B ($P_X = 3$)

If we assume all three channels are active with $P_X = 3$:

$$3H - 13 = 3 \implies 3H = 16 \implies H \approx 5.33$$

This yields a negative power for Channel 3 ($P_3 = 5.33 - 7 = -1.67$), which is impossible. Thus, Channel 3 must be deactivated ($\mathbf{P_3 = 0}$).

1. **Recalculate assuming only Channels 1 and 2 are active:**

$$(H - \sigma_1^2) + (H - \sigma_2^2) = P_X$$
$$(H - 2) + (H - 4) = 3$$
$$2H - 6 = 3 \implies 2H = 9 \implies \mathbf{H = 4.5}$$

2. **Calculate individual power allocations:**
   - $P_1 = 4.5 - 2 = \mathbf{2.5}$
   - $P_2 = 4.5 - 4 = \mathbf{0.5}$
   - $P_3 = \mathbf{0}$

   Since the water level $H = 4.5 \leq \sigma_3^2 = 7$, keeping Channel 3 inactive is mathematically consistent.

3. **Calculate total system capacity C:**

$$C = \frac{1}{2}\left[\log_2\left(1 + \frac{2.5}{2}\right) + \log_2\left(1 + \frac{0.5}{4}\right) + \log_2(1)\right]$$
$$C = \frac{1}{2}\left[\log_2(2.25) + \log_2(1.125)\right]$$
$$C \approx \frac{1}{2}\left[1.1699 + 0.1699\right] = \mathbf{0.6699} \text{ bits}$$