2nd Midterm in Information Theory (TOR III)

(May 26, 2026)

1. Let the probability transition matrix of a Markov process be given by,

$$\Pi = \begin{pmatrix} 0.8 & 0.2 & 0 \\ 0 & 0.7 & 0.3 \\ 0.1 & 0 & 0.9 \end{pmatrix}$$

   That is, there are three states S_1, S_2, S_3 and the entries $\pi_{i,j}$ denotes the probability of going to state S_j from state S_i.

   (a) Calculate the stationary distribution of this process.

   (b) Find the entropy rate H_∞.

   (c) Denote the states S_1, S_2, S_3 by A, B, C respectively. Consider the extended alphabet by looking at two states so that the alphabet now consists of AA, AB, AC, BB, BC, CC. What is the average number of bits per symbol in this case if Huffman coding is used? Hint: Do not forget the stationary distribution of the symbols A, B, C, e.g. $P(AB)$ means that you are in state A and going to B.

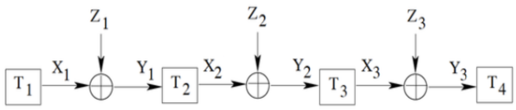
2. A Professor gives varying versions of an oral examination in assembly line fashion, with students taking the exam one after the other. Each version of the exam may be categorized as difficult, normal or easy. After a difficult exam, the next exam will be difficult with probability 0.2, will be normal with probability 0.5, and will be easy with probability 0.3. Similarly, after normal and easy exams, these probabilities are 0.5, 0.25 and 0.25 respectively.

   (a) What is the steady state distribution for this Markov chain?

   (b) Find the entropy rate H_∞ for this Markov chain (you might find the formula $H_\infty = \sum_i P(S_i)H(S_i)$ useful).

3. Consider the following communication system transmitting from Terminal T1 to Terminal T4 through 3 consecutive Gaussian channels with additive noise:

   The power of the noise signal Z_i equals N_i, for $i = 1, 2, 3$. Terminal i, where $i = 2, 3, 4$, receives the signal Y_{i-1}, decodes it, and, if appropriate, encodes it for further transmission. The power of the signal X_i is P_i, for $i = 1, 2, 3$, and the power constraint is $P_1 + P_2 + P_3 \leq P$.



(a) In terms of P_i and N_i, determine the capacity C_i of the channel

$$T_i \to T_{i+1}, \qquad i = 1, 2, 3.$$

(b) Suppose for the moment that P_1, P_2, P_3 are fixed. What is the Shannon capacity C of the communication from T_1 to T_4? Express your answer in terms of C_i.

(c) Determine the optimal power allocation P_1, P_2, P_3. What is the corresponding value of C? Think about the problem as a series of channels. Would it be reasonable to have some $P_i = 0$? The result from part (b) may be useful.

4. This problem concerns the error-correcting capabilities of linear codes.

   (a) For any block code with minimum Hamming distance at least $2t + 1$ between codewords, show that:

   $$2^{n-k} \geq 1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{t},$$

   where n is the length and k the dimension of a linear code.

   (b) A team of students is trying to design a $[20, 16]$ linear block code for single error correction. Explain whether they are likely to succeed or not.

5. (30) A linear code with parameters $[N, K, d]$ (denoting the length, dimension and minimum distance respectively) has a capability of correcting t errors.

   (a) (15) Show that if $d = 2t+1$ then this code can correct up to t errors. Hint: Use the triangle inequality and show that the distance of the received codeword to other codewords (different form the sent one) is larger than the distance to the originally sent codeword. (**Hint**: If u is sent and v received with no more than t errors, show that $d(u, v) < d(u, u')$.

   (b) (5) Is it possible that this code can sometimes correct more than t errors when the syndrome decoding is used, explain.

   (c) (10) Specify the generator matrix of the code of length $N = 6$ given by the following parity check equations (using the standard form for H and G): $u_1 + u_3 + u_5 = 0$; $u_2 + u_3 + u_4 + u_6 = 0$. Decide what is its minimum distance by considering the parity check matrix built using the above equations.

1 Solutions

1.1 Solution:P-1

a) Using $\mu\Pi = \mu$, where $\mu = (\mu_1, \mu_2, \mu_3)$, and $\mu_1 + \mu_2 + \mu_3 = 1$ one gets

$$(\mu_1, \mu_2, \mu_3) = \left(\frac{3}{11}, \frac{2}{11}, \frac{6}{11}\right).$$

b) Assuming the process is ergodic thus applying the formula $H_\infty = \sum_i P(S_i)H(S_i)$, where S_i denote the states of the Markov process.

The entropy rate $H_\infty = \sum_i P(S_i)H(S_i)$ can be written as $\sum_i \mu_i \sum_j P_{i,j} \ln P_{i,j}$ which then gives

$$H_\infty = (3/11)(-0.8\log_2 0.8 - 0.2\log_2 0.2) + (2/11)(-0.7\log_2 0.7 - 0.3\log_2 0.3) + \\ + (6/11)(-0.9\log_2 0.9 - 0.1\log_2 0.1) = 0.613.$$

c) The probabilities of the symbols are

$$\begin{aligned} P(AA) &= 0.8 \cdot 3/11 = 24/110 \\ P(AB) &= 0.2 \cdot 3/11 = 6/110 \\ P(BB) &= 0.7 \cdot 2/11 = 14/110 \\ P(BC) &= 0.3 \cdot 2/11 = 6/110 \\ P(CC) &= 0.9 \cdot 6/11 = 54/110 \\ P(CA) &= 0.1 \cdot 6/11 = 6/110. \end{aligned}$$

Calculation of Average Bits per Extended Symbol:

Symbol	**Calculation**	**Probability**	**Huffman Code**	**Length**
CC	$\frac{6}{11} \times 0.9$	54/110	0	1
AA	$\frac{3}{11} \times 0.8$	24/110	10	2
BB	$\frac{2}{11} \times 0.7$	14/110	110	3
BC	$\frac{2}{11} \times 0.3$	6/110	1110	4
AB	$\frac{3}{11} \times 0.2$	6/110	11110	5
CA	$\frac{6}{11} \times 0.1$	6/110	11111	5

$$L_{\text{avg}} = \frac{1}{110}[54(1) + 24(2) + 14(3) + 6(4) + 6(5) + 6(5)] = \frac{208}{110} \approx 1.891 \text{ bits}$$

1.2 Solution:P-2

Let's call the states "hard exam", "medium exam", "easy exam" respectively 1, 2, 3.

From the information given to us in the statement of the problem we deduce that the transition probability matrix is:

$$M = \begin{bmatrix} 0.2 & 0.5 & 0.3 \\ 0.5 & 0.25 & 0.25 \\ 0.5 & 0.25 & 0.25 \end{bmatrix}$$

a) To get the steady state distribution we have to solve $(v_1, v_2, v_3)M = (v_1, v_2, v_3)$ and $v_1 + v_2 + v_3 = 1$. Multiplying the matrix we get the following system of linear equations:

$$\begin{aligned} v_1 + v_2 + v_3 &= 1 \\ \frac{1}{5}v_1 + \frac{1}{2}v_2 + \frac{1}{2}v_3 &= v_1 \\ \frac{1}{2}v_1 + \frac{1}{4}v_2 + \frac{1}{4}v_3 &= v_2 \\ \frac{3}{10}v_1 + \frac{1}{4}v_2 + \frac{1}{4}v_3 &= v_3 \end{aligned}$$

$$\begin{aligned} &1)\ v_1 + v_2 + v_3 = 1 \\ &2)\ \frac{1}{5}v_1 + \frac{1}{2}v_2 + \frac{1}{2}v_3 = v_1 \implies -\frac{4}{5}v_1 + \frac{1}{2}(v_2 + v_3) = 0 \\ &3)\ \frac{1}{2}v_1 + \frac{1}{4}v_2 + \frac{1}{4}v_3 = v_2 \implies \frac{1}{2}v_1 - \frac{3}{4}v_2 + \frac{1}{4}v_3 = 0 \end{aligned}$$

Substitute $v_2 + v_3 = 1 - v_1$ into (2):

$$-\frac{4}{5}v_1 + \frac{1}{2}(1 - v_1) = 0 \implies -8v_1 + 5 - 5v_1 = 0 \implies v_1 = \frac{5}{13}$$

Substitute $v_1 = \frac{5}{13}$ into (1) and (3):

$$v_2 + v_3 = \frac{8}{13} \quad \text{and} \quad \frac{5}{26} - \frac{3}{4}v_2 + \frac{1}{4}v_3 = 0$$

Solving the resulting system yields:

$$v_1 = \frac{5}{13}, \quad v_2 = \frac{9}{26}, \quad v_3 = \frac{7}{26}$$

The solution, and hence the steady state distribution, is $v_1 = \frac{5}{13}$, $v_2 = \frac{9}{26}$ and $v_3 = \frac{7}{26}$.

b) Using the formula given to us in the problem, we have:

$$
\begin{aligned}
H_\infty &= \sum_i P(S_i)H(S_i) \\
&= \frac{5}{13}\left(\frac{1}{5}\log 5 + \frac{1}{2}\log 2 + \frac{3}{10}\log\frac{10}{3}\right) + \frac{9}{26}\left(\frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{4}\log 4\right) + \\
&\quad + \frac{7}{26}\left(\frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{4}\log 4\right) \\
&= 1.4944.
\end{aligned}
$$

1.3 Solution:P-3

1. The capacity of the Gaussian channel $T_i \to T_{i+1}$ is

$$C_i = \frac{1}{2}\log\left(1 + \frac{P_i}{N_i}\right).$$

2. Since the communication passes through all three channels consecutively, the overall capacity is determined by the bottleneck channel. Hence,

$$C = \min_{1 \le i \le 3} C_i.$$

3. To maximize the minimum capacity, we require

$$C_1 = C_2 = C_3.$$

Therefore,

$$\frac{P_1}{N_1} = \frac{P_2}{N_2} = \frac{P_3}{N_3}.$$

Using the total power constraint,

$$P = P_1 + P_2 + P_3,$$

we obtain

$$P = P_1 + P_1\frac{N_2}{N_1} + P_1\frac{N_3}{N_1}.$$

Hence,

$$P_1 = P\frac{N_1}{N_1 + N_2 + N_3}.$$

Similarly,

$$P_i = P\frac{N_i}{N_1 + N_2 + N_3}, \qquad i = 1, 2, 3.$$

Substituting this into the capacity expression gives

$$C = \frac{1}{2}\log\left(1 + \frac{P}{N_1 + N_2 + N_3}\right).$$

1.4 Solution: P-4

a)

An $[n, k]$ block code can represent in its parity bits at most 2^{n-k} patterns, and these must cover all the error cases we wish to correct, as well as the one case with no errors. When the minimum Hamming distance is $2t + 1$, the code can correct up to t errors. The number of ways in which the transmission can have $0, 1, 2, \ldots, t$ errors which then gives the result.

b)

A single error correcting code must be able to uniquely identify $n+1$ patterns (n error patterns and one without any errors), see also part a). So 2^{n-k} must be $\geq n + 1$. That is not true when $n = 20$ and $k = 16$.

1.5 Solution: P-5

a)

See the lecture notes.

b)

When filling the syndrome table with different errors with weight at most t, there might be remaining entries among 2^{N-K} entries in total.

C)

The generator matrix and minimum distance of the code are determined below.

1. Parity Check Matrix (H)

From the given equations, we construct the parity check matrix H of size 2×6:

$$H = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

Notice that columns 5 and 6 form a 2×2 identity matrix (I_2). We can express H in the standard form $H = [A \mid I_2]$:

$$H = \left(\begin{array}{cccc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array} \right)$$

Where matrix A is defined as:

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

2. Generator Matrix (G)

For a parity check matrix in the standard form $H = [A \mid I_{n-k}]$, the corresponding standard form for the generator matrix is $G = [I_k \mid A^T]$ over the binary field $\mathbb{F}_2$ (where $-A^T = A^T$):

$$G = \left(\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

3. Minimum Distance (d_{min})

The minimum distance of a linear code equals the minimum number of linearly dependent columns in its parity check matrix H.

Let the columns of H be denoted as c_1 through c_6:

$$c_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad c_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad c_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad c_5 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad c_6 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We can identify identical columns within the matrix:

- $c_1 = c_5 \implies c_1 + c_5 = 0$
- $c_2 = c_4 \implies c_2 + c_4 = 0$

Since there exist pairs of identical columns, the smallest number of linearly dependent columns is 2.

Therefore, the minimum distance of the code is $d_{min} = 2$.