Solutions - The second exam in TOR III - 02.07.2020

Problem 1. The two random variables X and Y take values in $x \in \{0, 1\}$ and $y \in \{0, 1, 2\}$, respectively. Their joint distribution function can be written as

$$P(x, y) = K \cdot (x + y).$$

(20 points)

(a) Compute K.
(b) Find $H(X)$ and $H(Y)$.
(c) Find $I(X; Y)$.

Solution:
(a) Setting up a table for $K \cdot (x + y)$ we have:

$K \cdot (x+y)$	$y = 0$	$y = 1$	$y = 2$
$x = 0$	0	K	2K
$x = 1$	K	2K	3K

Since we also need that $\sum p(x, y) = 1$, we get $9K = 1$, and so $K = \frac{1}{9}$.

(b) We first derive the probabilities for $p(x)$ and $p(y)$ by summing the columns and rows respectively, i.e.:

$K \cdot (x+y)$	$y = 0$	$y = 1$	$y = 2$	$p(x)$
$x = 0$	0	1/9	2/9	3/9
$x = 1$	1/9	2/9	3/9	6/9
$p(y)$	1/9	3/9	5/9	

Now, we can compute:

$$H(X) = \frac{3}{9} \log_2(\frac{9}{3}) + \frac{6}{9} \log_2(\frac{9}{6}) = 0.918$$

and

$$H(Y) = \frac{1}{9} \log_2(\frac{9}{1}) + \frac{3}{9} \log_2(\frac{9}{3}) + \frac{5}{9} \log_2(\frac{9}{5}) = 1.35$$

.

(c) The joint entropy is $H(X,Y) = 0 + \frac{1}{9}\log_2(\frac{9}{1}) + \frac{1}{9}\log_2(\frac{9}{1}) + \frac{2}{9}\log_2(\frac{9}{2}) + \frac{2}{9}\log_2(\frac{9}{2}) + \frac{3}{9}\log_2(\frac{9}{3}) = 15/9\log 3 - 4/9$, and so:

$$I(X;Y) = H(X) + H(Y) - H(X,Y) = 0.072.$$

Problem 2 A memoryless source produces symbols in $x \in \{0,1,2,3,4\}$ according to the probabilities: **(15 points)**

$$\begin{pmatrix} x \\ p(x) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{6} & \frac{1}{6} & \frac{1}{2} \end{pmatrix}$$

(a) Compute the entropy of the source.

(b) Construct an optimal binary source code for the source (hint:Huffman). What is the expected codeword length? Compare with the entropy of the source.

Solution:

(a) The entropy of the source is $H(X) = H(1/12, 1/12, 1/6, 1/6, 1/2) = \frac{1}{12}\log_2(12) + \frac{1}{12}\log_2(12) + \frac{1}{6}\log_2(6) + \frac{1}{6}\log_2(6) + \frac{1}{2}\log_2(2) = 1.959$

(b) Huffman coding is optimal for a symbol-by-symbol coding with a known input probability distribution, which is the case here. The Huffman tree for this distribution is (see the lecture slides, we take the two smallest values and add them to get the next column)

0	$\frac{1}{12}$				
1	$\frac{1}{12}$	$\frac{1}{6}$			
2	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$		
3	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$	
4	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1

Going backwards (with respect to columns, see the lecture slides), and adding 0 when we "go up", and 1 when we "go down" with the red squares, and doing nothing when we have a white square, we get the encoding:

0	0000
1	0001
2	001
3	01
4	1

The expected codeword length is $E[L] = 4\frac{1}{12} + 4\frac{1}{12} + 3\frac{1}{6} + 2\frac{1}{6} + 1\frac{1}{2} = 2$, where L is a random variable that denotes the length of the coded symbols.

Problem 3. Let $C(x) = \frac{1}{2}\log_2(1+x)$ be the channel capacity of a Gaussian channel with signal to noise ratio x, which we usually denoted $x = \frac{P}{N}$. Show that

$$C\left(\frac{P_1}{N}\right) + C\left(\frac{P_2}{P_1+N}\right) = C\left(\frac{P_1+P_2}{N}\right),$$

which essentially means that 2 independent users can send information as if they had pooled (joined) their power. **(10 points)**

Solution:

$$\begin{aligned}
C\left(\frac{P_1+P_2}{N}\right) &= \frac{1}{2}\log_2\left(1+\frac{P_1+P_2}{N}\right) \\
&= \frac{1}{2}\log_2\left(\frac{N+P_1+P_2}{N}\right) \\
&= \frac{1}{2}\log_2\left(\frac{N+P_1+P_2}{P_1+N}\cdot\frac{P_1+N}{N}\right) \\
&= \frac{1}{2}\log_2\left(\frac{N+P_1+P_2}{N+P_1}\right) + \frac{1}{2}\log_2\left(\frac{P_1+N}{N}\right) \\
&= C\left(\frac{P_2}{P_1+N}\right) + C\left(\frac{P_1}{N}\right)
\end{aligned}$$

Problem 4. This problem concerns linear codes built form parity check equations. **(15 points)**

a) Let the following parity check equations be used to specify a binary linear code of length 7, thus its codewords $(c_1, \ldots, c_7)$ satisfy:

$$\begin{aligned}
c_1 \oplus c_2 \oplus c_5 &= 0 \\
c_2 \oplus c_3 \oplus c_6 &= 0 \\
c_1 \oplus c_2 \oplus c_3 \oplus c_7 &= 0 \\
c_2 \oplus c_5 \oplus c_6 \oplus c_7 &= 0
\end{aligned}$$

How many codewords such a code has and what is its minimum distance ?

Solution: The last equation is linearly dependent, so there are 3 lin. independent parity check equations. Therefore, the code has $2^{7-3} = 2^4 = 16$ codewords. Its minimum distance is 1 since $c = (0,0,0,1,0,0,0)$ is a valid codeword (the 4th column of H is an all-zero vector).

b) Write the generator matrix of this code in systematic form (using H from a)).

Solution: The generator matrix is given as

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Problem 5: In Figure below the plan for a house is given. Some of the doors is one-way while others can be used both ways, this is marked in the plan by arrows. A cat walks around in the house at random. When the cat is in a room it can either stay or leave by some of the outgoing doors. It chooses among the alternatives randomly with equal probability. Since there are ants in room 3, the cat will leave that room directly and not stay. **(20 points)**

1. What is the steady state distribution for what room the cat is in.

   Viewing the rooms as states in a Markov chain, we get the state transition matrix

   $$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

   Then, using $\mathbf{w}P = \mathbf{w}$ and $\sum_{i=1}^{4} \mathbf{w}_i = 1$ we get $\mathbf{w} = (\frac{6}{25} \; \frac{2}{5} \; \frac{6}{25} \; \frac{3}{25})$.

2. Find the entropy rate H_∞ (assuming the process is ergodic thus applying the formula $H_\infty = \sum_i P(S_i)H(S_i)$) for the cat's walk.

$$H_\infty = \sum_i P(S_i)H(S_i) = \frac{6}{25}H(\frac{1}{3},\frac{1}{3},\frac{1}{3}) + \frac{2}{5}H(\frac{1}{2},\frac{1}{2}) + \frac{6}{25}H(\frac{1}{2},\frac{1}{2}) + \frac{3}{25}H(\frac{1}{3},\frac{1}{3},\frac{1}{3}) = 1.21.$$