MATHEMATICAL METHODS

LECTURE 1

1. Logic and set theory

The basic blocks of logical deductions are *statements* (or *propositions*), which can be symbolically denoted by $p, q, r, \ldots$ or $A, B, C, \ldots$. For a statement we consider every assertion which is *true* or *false*, but *never both*. It is customary to denote by 1 if the statement is true, and by 0 if it is false. Also, p or n for true or false, respectively, where this notation is more common if we denote statements by $A, B, C, \ldots$.

Example 1. The statement "Five is an odd number" is true. The statement "Eight is a prime number" is false. The sentence "What's the weather today?" is not a statement.

1.1. Operations with propositions.

1.1.1. *Negation.* Let p be a statement. Negation of p is denoted by $\neg p$ and we read "not p". Negation of p is true precisely when p is false. Therefore, the following is the *truth table* for $\neg$.

p	$\neg p$
1	0
0	1

1.1.2. *Conjunction.* From two statements p and q we can construct another statement such that we connect p and q with a word *and*. Then "p and q" is the conjunction of statements p and q, which we denote by $p \wedge q$. Conjunction of p and q is true precisely when p and q are true:

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

1.1.3. *Disjunction.* From two statements p and q we can construct another statement such that we connect p and q with a word *or*. Then "p or q" is the disjunction of statements p and q, which we denote by $p \vee q$. Disjunction of p and q is true precisely when at least one of p, q is true:

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

1.1.4. *Implication.* If two statements p and q are in relation "if p, then q" or "from p it follows q", then we have the implication of p and q, where p is the *starting condition* (or *antecedent* or *hypothesis*) and q is the *resulting outcome* (or *consequent* or *conclusion*). We denote this by $p \implies q$ and read it also as "p is a sufficient condition for q" or "q is a necessary condition for p". The implication of p and q is false precisely when p is true and q is false:

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p	q	$p \implies q$
1	1	1
1	0	0
0	1	1
0	0	1

1.1.5. *Equivalence.* If two statements p and q are in relation "p if and only if q", then we have the equivalence of p and q. We denote this by $p \iff q$ and read it also as "p is a necessary and sufficient condition for q". The equivalence of p and q is true precisely when p and q are both true or both false:

p	q	$p \iff q$
1	1	1
1	0	0
0	1	0
0	0	1

Obviously we have $(p \iff q) \iff ((p \implies q) \wedge (q \implies p))$.

1.2. **Some properties of these operations.** The following is a list of the most important properties.

(1) $\neg(\neg p) \iff p$;
(2) Commutativity: $p \wedge q \iff q \wedge p$ and $p \vee q \iff q \vee p$;
(3) Associativity: $(p \wedge q) \wedge r \iff p \wedge (q \wedge r)$ and $(p \vee q) \vee r \iff p \vee (q \vee r)$;
(4) Distributivity: $p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$ and $p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$;
(5) De Morgan's laws: $\neg(p \wedge q) \iff \neg p \vee \neg q$ and $\neg(p \vee q) \iff \neg p \wedge \neg q$;
(6) Contraposition: $(p \implies q) \iff (\neg q \implies \neg p)$;
(7) Material implication: $(p \implies q) \iff (\neg p \vee q)$;
(8) Contradiction: $\neg(p \wedge \neg p)$ is always true, i.e., is a *tautology*.

Example 2. Prove (6) and (7). We are using truth tables:

p	q	$\neg p$	$\neg q$	$p \implies q$	$\neg q \implies \neg p$	$\neg p \vee q$
1	1	0	0	1	1	1
1	0	0	1	0	0	0
0	1	1	0	1	1	1
0	0	1	1	1	1	1

The last three columns are the same.

1.3. **Proof of implication by contradiction.** By the material implication we have $(p \implies q) \iff (\neg p \vee q)$. Therefore,

$$\neg(p \implies q) \iff \neg(\neg p \vee q) \iff p \wedge \neg q$$

by the contraposition and De Morgan's law. This enables us to develop the following strategy to prove implications:

(1) We want to prove $p \implies q$, that is, the statement $p \implies q$ is true;
(2) We assume that p and $\neg q$ are true;
(3) With a correct usage of mathematical laws we arrive to a contradiction, that is, both statements r and $\neg r$ are correct.
(4) This means that $p \wedge \neg q$ is false. By the above we then have that $\neg(p \implies q)$ is false, so $p \implies q$ must be true.

Example 3. Prove the following: If n^2 is an even integer, then n is also an even integer. Assume that n^2 is even and n is odd. From the latter, there exists $k \in \mathbb{Z}$ such that $n = 2k + 1$. From the former we have $2 \mid n^2$. Now,

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

and $2k^2 + 2k \in \mathbb{Z}$, so $2 \nmid n^2$. We obtained $2 \mid n^2$ and $2 \nmid n^2$. This is contradiction!

1.4. **Quantifiers.** Sometimes we want to know if certain property holds for all elements of some set or only for at least one element, where a set is a collection of some elements. We write such statements with the help of *quantifiers*.

(1) Quantifier $\forall$. We symbolically write the statement that p holds *for every* element x as $\forall x.\, p(x)$ and say "for every x, p holds".
(2) Quantifier $\exists$. We symbolically write the statement that p holds *for at least one* element x as $\exists x.\, p(x)$ and say "there exists x such that p holds".
(3) Quantifier $\exists!$. We symbolically write the statement that p holds *for exactly one* element x as $\exists! x.\, p(x)$ and say "there exists a unique x such that p holds".

The negation of quantifiers:

$$\neg(\forall x.\, p(x)) \iff \exists x.\, \neg p(x), \quad \neg(\exists x.\, p(x)) \iff \forall x.\, \neg p(x).$$

So, the negation "flips" the quantifier type *and* negates the inner statement. For *nested* quantifiers we have

$$\neg(\forall x.\, \exists y.\, p(x,y)) \iff \exists x.\, \forall y.\, \neg p(x,y).$$

In many cases we are specifying the *domain of discourse* D: $\forall x \in D.\, p(x)$ and $\exists x \in D.\, p(x)$. This notation is known as *restricted* (or *bounded*) *quantification*. We have the following:

$$(\forall x \in D.\, p(x)) \iff \forall x.\, (x \in D \implies p(x)), \quad (\exists x \in D.\, p(x)) \iff \exists x.\, (x \in D \wedge p(x)).$$