MATHEMATICAL METHODS

LECTURE 2

1. Set theory

A *set* is a collection of things which we call them *elements*. Usually we denote sets with upper letters (e.g., A, B, X, Y) while elements are denoted by lower letters (e.g., a, b, x, y). If a is an element of A, then we denote this as $a \in A$. If a is not an element of A, then we denote this as $a \notin A$. The *empty set* is a set with no elements, and we denote it as $\{\}$ or $\varnothing$. A set is *finite* if it contains finitely many elements. The number of elements (cardinality) of a finite set A is denoted by $|A|$.

Example 1. Three simple examples:

(1) $A = \{1, 2, 3, 4\}$ is a set with four elements;
(2) The set B of all positive integers divisible by 5: $B = \{5, 10, 15, 20, \ldots\} = \{n \in \mathbb{N} \colon 5 \mid n\}$;
(3) Elements of sets can also be sets, e.g., $\{1, \{1, 2\}\}$ is a set with two elements, number 1 and a set $\{1, 2\}$.

We say that A is a *subset* of a set B if and only if every element of A is also an element of B:

$$A \subseteq B \iff \forall x \in A.\, x \in B.$$

Two sets A and B are *equal*, writing as $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$. If $A \subseteq B$ and $A \neq B$, then A is a *proper subset* of B, writing as $A \subset B$. We have the following properties:

(1) Reflexivity: $A \subseteq A$;
(2) Antisymmetry: $A \subseteq B \wedge B \subseteq A \implies A = B$;
(3) Transitivity: $A \subseteq B \wedge B \subseteq C \implies A \subseteq C$.

Example 2. Three simple examples:

(1) $\{\} \subseteq A$ for every set A;
(2) $\{1, 2, 3, 4\} = \{1, 2, 3, 2, 4, 4, 2\}$: a set is equal even if the elements duplicate;
(3) $\{n \in \mathbb{N} \colon n \text{ is a prime number}\} \subset \{n \colon n \in \mathbb{N}\} \subset \{n \colon n \in \mathbb{Z}\}$.

1.1. Operations with sets.

It is useful to visualize sets (and operations with them) by using *Venn diagrams*, see `https://en.wikipedia.org/wiki/Venn_diagram`.

1.1.1. *Intersection.*

$$A \cap B = \{x \colon x \in A \wedge x \in B\}.$$

If $A \cap B = \varnothing$, then we say that A and B are *disjoint sets*. Some properties:

(1) Commutativity: $A \cap B = B \cap A$;
(2) Associativity: $(A \cap B) \cap C = A \cap (B \cap C)$;
(3) $A \cap A = A$;
(4) $A \cap \varnothing = \varnothing$.

1.1.2. *Union.*

$$A \cup B = \{x \colon x \in A \vee x \in B\}.$$

Some properties:

(1) Commutativity: $A \cup B = B \cup A$;
(2) Associativity: $(A \cup B) \cup C = A \cup (B \cup C)$;
(3) $A \cup A = A$;
(4) $A \cup \varnothing = A$.

Distributive laws for the intersection and the union:

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(1) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$;
(2) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

1.1.3. *Set difference.*

$$A \setminus B = A - B = \{x \colon x \in A \wedge x \notin B\}.$$

Some properties:

(1) $A \setminus B \neq B \setminus A$: this operation is *not* commutative;
(2) $(A \setminus B) \setminus C \neq A \setminus (B \setminus C)$: this operation is *not* associative;
(3) $A \setminus A = \varnothing$;
(4) $A \setminus \{\} = A$.

1.1.4. *Complement.* The complement of a set A, usually denoted by A^c, is the set of elements not in A. If $\mathcal{U}$ is the *universal set* (the one that contains "all" elements), then

$$A^c = \{x \colon x \in \mathcal{U} \wedge x \notin A\} = \mathcal{U} \setminus A.$$

Some properties:

(1) $\mathcal{U}^c = \varnothing$;
(2) $\varnothing^c = \mathcal{U}$;
(3) $(A^c)^c = A$;
(4) $A \cap A^c = \varnothing$;
(5) $A \cup A^c = \mathcal{U}$;
(6) De Morgan's laws: $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$.

1.1.5. *Power set.*

$$\mathcal{P}(A) = \{X \colon X \subseteq A\}.$$

So, the power set $\mathcal{P}(A)$ of a given set A is the new set that contains all subsets of A. If A is finite, then $|\mathcal{P}(A)| = 2^{|A|}$.

Example 3. Let $A = \{a, b\}$. Then $\mathcal{P}(A) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$.

1.1.6. *Cartesian product. Ordered pair* (a, b) is defined as $(a, b) = \{\{x\}, \{x, y\}\}$. It follows that $(x, y) = (x', y')$ if and only if $x = x'$ and $y = y'$. Therefore, $(x, y) \neq (y, x)$ for $x \neq y$. The *cartesian product* $A \times B$ of sets A and B is defined as

$$A \times B = \{(x, y) \colon x \in A \wedge y \in B\}.$$

Some properties:

(1) $A \times B \neq B \times A$;
(2) $(A \times B) \times C = A \times (B \times C)$;
(3) $A \times B = \varnothing \iff A = \varnothing \vee B = \varnothing$.

Generalization:

$$A_1 \times A_2 \times \cdots \times A_n = \{(x_1, x_2, \ldots, x_n) \colon x_1 \in A_1 \wedge x_2 \in A_2 \wedge \cdots \wedge x_n \in A_n\}.$$