**Analysis I**
UP - FAMNIT
Fall 2026 October 07-08, 2026

# Tutorial 1

1. Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ and denote the successor of $n$ by $n^+$.
   (a) Write the number 4 in terms of the successor of 1, i.e., by using symbols 1, the successor operation, and parenthesis.
   (b) Addition $m + n$ is defined by applying the successor operation to $m$ exactly $n$ times. Use this definition to verify 1+4=5.

2. Is $\mathbb{N}$ closed under subtraction? Is $\mathbb{Z}$ closed under division by nonzero integers? Justify your answers. (*Left as an exercise*)

3. Recall that, for $a, b \in \mathbb{Z}$, we define $a < b$ if $b = a + n$ for some $n \in \mathbb{N} = \{1, 2, \ldots\}$. Using this definition, prove:
   (a) If $a < b$ and $b < c$, then $a < c$.
   (b) There is no integer $x$ such that $a < x < a + 1$.

   (*Left as an exercise*)

4. Prove the following statements for every positive natural number $n$:
   (a) $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$. (*Left as an exercise*)
   (b) $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$.

5. Prove that for any $n \in \mathbb{N}$, the integer $11^n - 4^n$ is a natural number multiple of 7.

6. Prove that $4^n + 15n - 1$ is divisible by 9 for all natural numbers $n$.

7. Prove the Bernoulli inequality: for every natural number $n$ and for every number $a > -1$ it holds that:
$$(1 + a)^n \geq 1 + na.$$

8. Prove that $n^{n+1} > (n + 1)^n$ for all natural numbers $n \geq 3$.

9. Prove that for all positive natural numbers $n$
$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$
   (*Left as an exercise*)

10. Prove that $3 + 11 + \cdots + (8n - 5) = 4n^2 - n$, for all positive integers $n$.
(*Left as an exercise*)

# Some Solutions

(3a) Write $b = a + m$ and $c = b + n$, where $m, n \in \mathbb{N}$. Then

$$c = a + (m + n).$$

Since $m + n \in \mathbb{N}$, we have $a < c$.

(3b) Suppose $a < x < a + 1$. Then $x = a + m$ and $a + 1 = x + n$ for some $m, n \in \mathbb{N}$. Hence

$$1 = m + n \geq 2,$$

a contradiction.