I-5 Real numbers

I-5-1 Dedekind cuts

Definition: A Dedekind cut for $\mathbb{Q}$ is a subset $A \subseteq \mathbb{Q}$ such that

1) $A \neq \emptyset$

2) $A \neq \mathbb{Q}$

3) if $x, y \in \mathbb{Q}$ with $x < y$ and $y \in A$ then $x \in A$ ("A is closed downwards")

4) if $x \in A$, there exists $y \in A$ s.t. $y > x$. (A does not have a greatest element)

Example 1) $A := \{q \in \mathbb{Q} \mid q < 3\}$ is a Dedekind cut.

1) $A \neq \emptyset$ (because $1 \in A$)

2) $A \neq \mathbb{Q}$ because $5 \notin A$.

3) If $x < y \in \mathbb{Q}$, $y \in A$, then $y < 3$. Because $x < y$, we have $x < 3$ by transitivity. Hence $x \in A$ also.

4) A does not have a largest element: $q < 3$

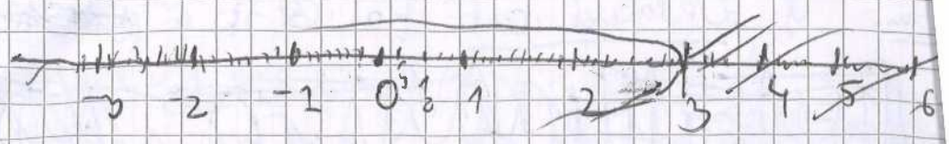
$\frac{q+3}{2} < 3$ because $\Leftrightarrow$ this is equiv. with $q + 3 < 6 \Leftrightarrow q < 3$

Note $\frac{q+3}{2} \in \mathbb{Q}$ because $\frac{q+3}{2} = \frac{1}{2} \cdot (q+3)$ ($\frac{1}{2} \in \mathbb{Q}$, $q+3 \in \mathbb{Q}$)

Hence $\frac{q+3}{2} \in A$

Also $q < \frac{q+3}{2}$ because $\Leftrightarrow 2q < q+3$

Visualization: $A = \{x \in \mathbb{Q} \mid x < 3\}$



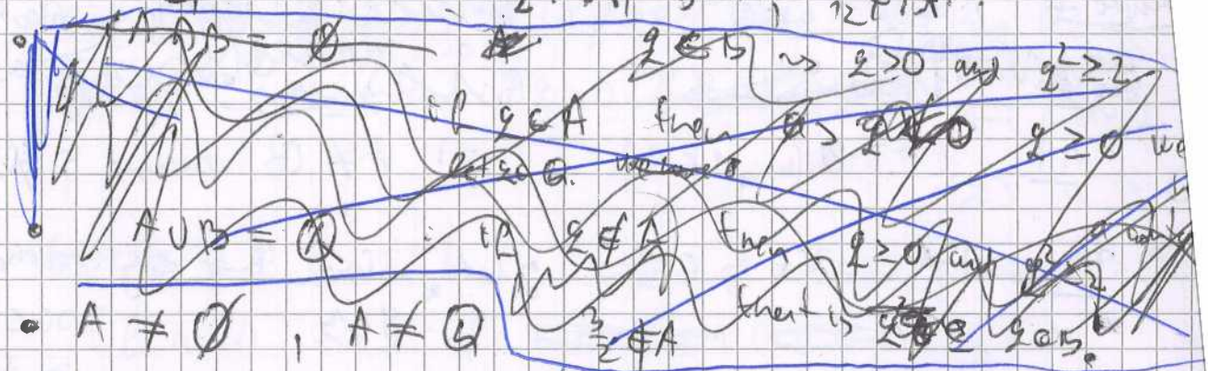


Another Example 2: $A := \{x \mid x \in \mathbb{Q}, x < 0 \text{ or } x^2 < 2\}$

Visualization:



$1 \in A$, $\frac{3}{2} \notin A$, $\frac{7}{5} \in A$, $\frac{17}{12} \notin A$, ... $A =$



- $A \neq \emptyset$, $A \neq \mathbb{Q}$
- if $x, y \in \mathbb{Q}$ and $y \in A$ and $x < y$, does $x \in A$?

Proof: if $y < 0$ then $x < y \Rightarrow$ (transitivity) $x < 0$, thus $x \in A$.

or $y \geq 0$, then $y^2 \leq 2$. if $x < 0$, we have $x \in A$.

if $x \geq 0$, then $x < y \Rightarrow x^2 <_{x<y} y^2$

(indeed, multiply by $x \geq 0$ we get $x^2 < yx < y^2$; $x < y$ multiplied by y)

hence $x^2 < 2$, thus $x \in A$.

- A does not have the greatest element.

Assume q_0 is the greatest element of A. Let us show that

If A is $1 \in q_0$, we know that $q_0 \geq 1 > 0$, hence $q_0^2 < 2$.

Consider the element $q_1 > q_0$ defined by $q_1 = \frac{2q_0+2}{q_0+2}$.

(Note that $q_0+2 > 0$, so it is safe to have q_0+2 in the denominator)

Observe that: $q_1 > q_0$. Indeed, $q_1 - q_0 = \frac{2q_0+2}{q_0+2} - q_0$

$$= \frac{2q_0+2}{q_0+2} = \frac{q_0(q_0+2)}{q_0+2} = \frac{(2q_0+2) - q_0(q_0+2)}{q_0+2} = \frac{2q_0+2-q_0^2-2q_0}{q_0+2}$$

$$= \frac{2-q_0^2}{q_0+2} > 0 \text{ as}$$

Observe that $q_1^2 < 2$. Indeed

$$q_1^2 - 2 = \left(\frac{2q_0+2}{q_0+2}\right)^2 - 2 = \frac{4q_0^2+8q_0+4-2(q_0^2+4q_0+4)}{(q_0+2)^2} = \frac{2(q_0^2-2)}{(q_0+2)^2}$$

$$< 0.$$

I-5-2

Definition: The set of real numbers is the set

$$\mathbb{R} := \{ A \mid A \text{ is a Dedekind cut of } \mathbb{Q} \}.$$

Each rational number $q \in \mathbb{Q}$ defines a Dedekind cut

$$q^* = \{ x \in \mathbb{Q} \mid x < r \}.$$ (Exercise check that q^* is a Dedekind cut)

From now on we may identify a rational number q with the corresponding Dedekind cut of $\mathbb{Q}$

Order: We say that $x < y$ if the corresponding

Subset, but not equal.

Defined cats satisfy $x \subsetneq y$. Write $x \leq y$ if

transitivity, trichotomy of $\leq$, reflexivity,
trichotomy: $x<y$ or $x=y$ or $x>y$ holding of $\leq$, $\geq$.

Definition:

A subset of the set of real numbers $S \subseteq \mathbb{R}$ is said to be bounded above, if there exists B s.t. $\forall s \in S$ one has that $s \leq B$.

A subset of the set of real numbers $S \subseteq \mathbb{R}$ is said to be bounded below, if there exists α s.t. $\forall s \in S$ one has that $s \geq \alpha$.

The greatest lower bound, of S is called the infimum of S. (the greatest means if l_0 is a lower bound for S, then $l_0 \leq l$). Notation $\inf S$

The smallest upper bound ... supremum of

Notation: $\sup S$.

INTERVALS:

$[a,b] := \{x \mid a \leq x \leq b\}$

$(a,b) := \{x \mid a < x < b\}$

ETC

Theorem: ("Supremum property") Every non-empty subset $S \subseteq \mathbb{R}$ bounded above admits a unique supremum.

Existence:

Proof: Let $\emptyset \neq S \subseteq \mathbb{R}$ be a non-empty subset of $\mathbb{R}$.

We identify $\Delta \in S$ with corresponding Dedekind cut for $\mathbb{Q}$

Let B be an upper bound for S.

Let $\mathcal{U} := \bigcup_{\Delta \in S} \Delta$. We show $\mathcal{U}$ is a Dedekind cut for $\mathbb{Q}$

① First, $\mathcal{U} \neq \emptyset$. Indeed, $S \neq \emptyset$ implies exists $\Delta_0 \in S$ and as Δ_0 is a Dedekind cut for $\mathbb{Q}$, we have that there exists $q \in \mathbb{Q}$ s.t. $q \in \Delta_0$. Then $q \in \mathcal{U}$, hence $\mathcal{U} \neq \emptyset$.

② Second, $\mathcal{U} \neq \mathbb{Q}$. ↯ Indeed, note $\mathcal{U} \subseteq B$ because each $\Delta \in S$ satisfies $\Delta \subseteq B$. But $\exists q \in \mathbb{Q}$ s.t. $q \notin B$.

Then necessarily $q \notin \mathcal{U}$.

(downward closed)

③ Assume $x, y \in \mathbb{Q}$ are such that $x < y$ and $y \in \mathcal{U}$.

As $y \in \mathcal{U}$, $\exists \Delta_0$ s.t. $y \in \Delta_0$. Then as Δ_0 is a Dedekind cut for $\mathbb{Q}$, we have $x \in \Delta_0$ and thus $x \in \mathcal{U}$

④ (no maximum element) Assume $y \in \mathcal{U}$. Then $y \in \Delta_0$ for some $\Delta_0 \in S$. As Δ_0 is a Dedekind cut for $\mathbb{Q}$, $\exists z \in \Delta_0$ s.t. $z > y$. But $\Delta_0 \subseteq \mathcal{U}$, thus $z \in \mathcal{U}$. ↯ Thus $\mathcal{U}$ doesn't possess a maximal element.

Uniqueness: If M_1, M_2 are both suprema for S, then $M_1 \leq M_2$, then M_2 both supremum thus $M_2 \leq M_1 \Leftrightarrow M_1 = M_2$

Exercise: Infimum property: Every non-empty subset $S \subseteq \mathbb{R}$ bounded below admits a unique infimum.

I-5-3 Basic operations

Addition If $\Delta \in \mathbb{R}$ and $B \in \mathbb{R}$ we define: $\Delta + B := \{a+b \mid a \in \Delta, b \in B\}$

We need to check that $\Delta + B$ is indeed a Dedekind cut for $\mathbb{Q}$ (take all sums $a+b$ with $a \in \Delta$ and $b \in B$)

$\alpha+\beta$ is a Dedekind cut for $\mathbb{Q}$.

$\alpha+\beta \neq \emptyset$. If $q \in \alpha$ and $r \in \beta$ then $q+r \in \alpha+\beta$

$\alpha+\beta \neq \mathbb{R}$. $\exists M_\alpha \in \mathbb{Q} \setminus \alpha$ (as $\alpha \neq \mathbb{Q}$)

$\exists M_\beta \in \mathbb{Q} \setminus \beta$ (as $\beta \neq \mathbb{Q}$)

note that $a \in \alpha \Rightarrow a < M_\alpha$, because α is downward

note that $b \in \beta \Rightarrow b < M_\beta$, because β is downward

Now $M_\alpha + M_\beta \in \mathbb{Q}$, but $M_\alpha + M_\beta \notin \alpha+\beta$ because

if $a \in \alpha$ and $b \in \beta$ are such $a+b < M_\alpha + M_\beta$.

Downward closed: if $x \in \alpha+\beta$ and $q \in \mathbb{Q}$ s.t. $q < x$

then $q \in \alpha+\beta$. ~~we need to show~~ $x \in \alpha+\beta \Rightarrow \exists a \in \alpha$ and $\exists b \in \beta$ s.t.

$x = a+b$. Assume $q < x = a+b \iff q-b < a$.

Let $a' = q-b$. Note that $a' \in \mathbb{Q}$ and $a' < a$, hence $a' \in \alpha$

Now $q = (q-b)+b = a' + b \in \alpha+\beta$.

No greatest element: $x \in \alpha+\beta$, write $x = a+b$ with $a \in \alpha$ and $b \in \beta$

$\exists a' \in \alpha$ s.t. $a' > a$. Note that $x' := a'+b \in \alpha+\beta$

and $x' > x$. Hence α does not have greatest element.

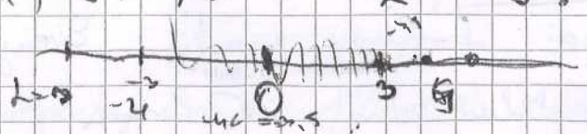
Properties: 1) $\alpha, \beta \in \mathbb{R}$ $\alpha+\beta = \beta+\alpha$ commutativity

2) $\alpha + 0 = 0 + \alpha = \alpha$. 0 neutral element for addition

Inverse for addition:

$-\alpha = \{q \in \mathbb{Q} \mid \exists r \notin \alpha$ s.t. $q < -r\}$.

Dedekind cut: for $\mathbb{Q}$!



- $-\alpha \neq \emptyset$: α is a Dedekind cut, $\exists n_0 \in \mathbb{Q} > \alpha$.

~~from above~~ set $q = -n_0 - 1 \in \mathbb{Q}$. Then $q \in -\alpha$, hence $-\alpha \neq \emptyset$

- $-\alpha \neq \mathbb{Q}$. Let $a_0 \in \alpha$.

we show
from $r - a_0 \notin -\alpha$. Indeed, suppose on the contrary that $-a_0 \in -\alpha$.

there exists $r \notin \alpha$ s.t. $-a_0 < -r \Rightarrow r < a_0$.

Since $a_0 \in \alpha$ and α is downward closed $\Rightarrow r \in \alpha$, which

is a contradiction with $r \notin \alpha$.

Downward closed: let $q \in -\alpha$, $p \in \mathbb{Q}$ s.t. $p < q$.

As $q \in -\alpha$, $\exists r \in \mathbb{Q} \setminus \alpha$ s.t. $q < -r$. Hence $p < -r$.

Or As $p < -r$, for the same $r \notin \alpha$, thus $p \in -\alpha$, hence downward closed

no greatest element.

let $q \in -\alpha$. $\exists r \in \mathbb{Q} \setminus \alpha$ s.t. $q < -r$. set

$s := \frac{q + (-r)}{2}$. then $q < s < -r$ (check.)

Because $s < -r$, and $r \notin \alpha$, $s \in -\alpha$ by definition.

As $s > q$, q is ~~not~~ greatest element

Property 3) $(-\alpha) + \alpha = \alpha + (-\alpha) = 0$.

Subtraction: $\alpha - \beta := \alpha + (-\beta)$

4) $\alpha < \beta$ $\forall \rho \in \mathbb{R}$ then $\alpha + \rho < \beta + \rho$.

~~Multiplication: Assume $\alpha > 0$ and $\beta > 0$.~~

Multiplication Let $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$ and $\beta > 0$

We define $\alpha \cdot \beta = \{q \in \mathbb{Q} \mid q \leq 0\} \cup \{a \cdot b \mid a \in \mathbb{Q} \cap \alpha, b \in \beta\}$ with $a > 0$

it is nonempty (since $-1 \in \alpha \cdot \beta$) and we $a > \alpha > 0$ $b > \beta > 0$ and $b > 0$

it is $\neq \mathbb{R}$ if α $\exists a$ s.t. $a \in \mathbb{Q} \setminus \alpha$ $\exists b \in \mathbb{Q} \setminus \beta$

$a \cdot b \in \mathbb{Q}$ $a \cdot b \notin \alpha \cdot \beta$ because $a \cdot b > 0$

and $a \cdot b > c \cdot b$ where $c \in \alpha$

Q

Archimedean closed $x, y \in$ ~~R~~ $x < y$ and $y \in \mathcal{L}\cdot\mathcal{D}$.

If $y = 0$ then clearly $x \in \mathcal{L}\cdot\mathcal{D}$.

If $y > 0$ then $y = a \cdot b$ for some $a \in \mathcal{L}$ and $b \in \mathcal{D}$.

$x < a \cdot b$ ~~$\Leftrightarrow$~~ If $x < 0$ we have $x \in \mathbb{Q}\,\mathcal{L}\mathcal{D}$.

Assume $0 < x < a \cdot b$. $\Rightarrow 0 < \frac{x}{b} < a$. $a' = \frac{x}{b}$ is rational and

$0 < a' < a$

hence $a' \in \mathcal{L}$ thus $x = a' \cdot b$ ~~$\in \mathbb{Q}$~~ $\in \mathcal{L}\cdot\mathcal{D}$

~~no~~ greatest element

Let $x \in \mathcal{L}\cdot\mathcal{D}$. If $x \leq 0$, then as $\mathcal{L} > 0$ and $\mathcal{D} > 0$

there exists $a \in \mathcal{L}$, $a > 0$ and $x \in \mathcal{D}$, $b > 0$ then $ab > 0$

If $x > 0$ then $x = ab$ for some $a \in \mathcal{L}$, $b \in \mathcal{D}$.

there exists $a' \in \mathcal{L}$ s.t. $a' > a$ so then $a'b \in \mathcal{L}\cdot\mathcal{D}$ and

$a'b > x$. ~~Not quite~~

~~Exc.~~ If multiplication by 0: $a \in \mathbb{R}$, by $a \cdot 0 = 0 \cdot a = 0$.

Multiplication $a > 0 > b$ then ~~$a \cdot b$~~ $-b > 0$

and we define $a \cdot b = -(a \cdot (-b))$.

Notation $x^2 = x \cdot x$, $x^n = \underbrace{x \cdot \ldots \cdot x}_{n \text{ times}}$

Properties: 1) Commutativity, 2) associativity 4.) 1 is neutral element

3) distributivity $+$, $\cdot$ 5) If $a < b$ and $c > 0$

then $ac < bc$ and if $c < 0$ then $ac > bc$

~~Archimedean~~ ~~Division~~ ~~$\mathcal{L} \neq \mathbb{T}$~~

6) existence of $\mathcal{L}^{-1}$, the multiplicative inverse. See exercise session

where $\mathcal{L} \neq 0$ $\mathcal{L}^{-1} \cdot \mathcal{L}$

Division. $\mathcal{L} / \mathcal{B} = \mathcal{L} \cdot \mathcal{B}^{-1}$.

I-5-4.

Theorem: Archemedean property: If $x, y \in \mathbb{R}$ and $x > 0$ there exists $n \in \mathbb{N}$ s.t. $n \cdot x > y$. (Alternate version as exercise)

Proof: $A := \{ n \cdot x \mid n \in \mathbb{N} \} \subset \mathbb{R}$.

Assume (on the contrary) y is an upper bound for A. As A is non-empty, and y is an upper bound, by the supremum property A has a supremum $\alpha := \sup A$.

Note that $\alpha - x < \alpha$ (because $x > 0$ and thus $-x < 0$ + α)

As $\alpha - x$ is smaller than the least upper bound of A, we have that $\alpha - x$ is not an upper bound for A. Hence $\exists m \in \mathbb{N}$ s.t. $mx > \alpha - x \iff mx + x > \alpha$.

$\Rightarrow (m+1)x > \alpha$. Now $(m+1)x \in A$ and this contradicts the fact that α is an upper bound for A. Thus y is not an upper bound for A, i.e. $\exists n \in \mathbb{N}$ s.t. $nx > y$.

Theorem: (Density of $\mathbb{Q}$ in $\mathbb{R}$)

Given $x, y \in \mathbb{R}$ with $x < y$, there exists $q \in \mathbb{Q}$ s.t. $x < q < y$.

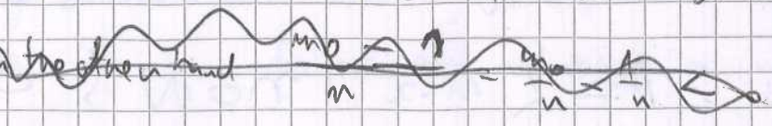
Proof: We have $y - x > 0$. By Archemedean property applied to $1 > 0$ there exists $n \in \mathbb{N}$ s.t. $n \cdot (y - x) > 1 \iff$

$ny - nx > 1 \iff ny > 1 + nx$.

Applying Archemedean property to nx (since $nx > 0$), $\exists m \in \mathbb{N}$ $m > nx$. Let $S := \{ m \in \mathbb{N} \text{ s.t. } m > nx \}$. By the well ordering of $\mathbb{N}$

We can choose m_0 to be the minimal element of S.

then $m_0 - 1 \leq nx < m_0$. Note that $q := \frac{m_0}{n}$ satisfies

$x < q$. 

on the other hand $m_0 \leq nx + 1 < ny$, thus $q = \frac{m_0}{n} <$

We have found a natural number q satisfying $x < q < y$.

I - 5 Existence Irrational numbers. (After next page)

We want to show that $\mathbb{Q} \subsetneq \mathbb{R}$.

Definition: We let a and $b \in \mathbb{Z}$. We say that $a \mid b$ if there exists an integer k s.t. $k \cdot a = b$. We say that $m \in \mathbb{Z}$ even if $2 \mid m$ and odd if $2 \nmid m$. (Exercise: Show m is odd then $m = 2$ for some k (does not divide))

Lemma: m is even then m^2 is even.

Proof: If m is even, then $m = 2k$, with $k \in \mathbb{Z}$. We have that $m^2 =$ thus $4 \mid m^2$.

(b) If m^2 is even, then m is even.

Proof: Assume (by contradiction) m is odd. then $m = 2k+1$. We have $m^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2 \cdot (2k^2 + 2k) + 1$ which is odd, a contradiction. Hence m is even.

(Pythagoras' theorem)

Theorem: Let $x := \sqrt{2}$. One has that $x \notin \mathbb{Q}$.

Proof: Assume by the contrary that $x \in \mathbb{Q}$. Then $x = \frac{m_0}{n_0}$ with $n_0 \in \mathbb{N}$ and $m_0 \in \mathbb{N}$. We have that $2 = x^2 = \frac{m_0^2}{n_0^2}$, that is $m_0^2 = 2n_0^2$. Let $S = \{n \in \mathbb{N} \mid \text{there exists } m \in \mathbb{Z} \text{ s.t. } m^2 = 2n^2\}$

Note $n_0 \in S$, hence S is non-empty. By well ordering of $\mathbb{N}$, we have that S has a minimal element, call it n_1. Then for some $m_1 \in \mathbb{Z}$ $m_1^2 = 2n_1^2$. $2 \mid m_1^2 \Rightarrow m_1$ is even. $m_1 = 2k$ $4k^2 = 2n_1^2$ $2k^2 = n_1^2$. But $k^2 < 2k^2 < n_1^2 \Rightarrow k < n_1$, and $k \in S$, a contradiction!

I-5.6. Existence of roots.

Theorem Let $x>0$. Let $n \in \mathbb{N}$. There exists a $y>0$ s.t. $y^n = x$.
(1) find a candidate for y

Proof: Let us define $E = \{ t \in \mathbb{R} \mid t>0 \text{ and } t^n < x \}$.

→ E is non-empty; as $\alpha = \frac{x}{1+x} \in E$. Indeed, $0 < \alpha < 1$.

$\Leftrightarrow \frac{x}{1+x} > 0$ because $x>0$ and $\alpha < 1$ because

$\frac{x}{1+x} < \frac{1+x}{1+x} = 1$. Moreover as $1+x>1$ and $x>0$

We have $\alpha = \frac{x}{1+x} < x$. Now $\alpha^n \leq \alpha$ because $\alpha < 1$

(exercise $n=1$ true, valid for k : $\alpha^{k+1} = \alpha^k \cdot \alpha \in E$ i.e. $\alpha^k \in E$)

Hence $\alpha^n \leq \alpha < x$ and thus $\alpha \in E$.

(2) $\forall t \in E$.

→ E is bounded above. Take $t_0 = 1+x$. If $t > 1+x$, then $t^n \geq t > 1+x > x$ so $t \notin E$. Contradiction, hence $1+x$ is an upper bound for E.

(exercise for $t \geq 1$)

By the supremum property, there is a supremum of E $y = \sup E$. (3) Let's see why

2.1 Show $y^n = x$. Assume $y^n < x$. Then

$$\left(y + \frac{1}{m}\right)^n - y^n \quad \text{for } 0 < \frac{1}{m} < 1.$$

$$= \frac{1}{m} \cdot \left(\left(y+\frac{1}{m}\right)^{n-1} + \left(y+\frac{1}{m}\right)^{n-2} y + \dots + \left(y+\frac{1}{m}\right) y^{n-2} + y^{n-1} \right) \quad \text{(exercise)}$$

$$\leq \frac{1}{m} \cdot n \cdot (1+y)^{n-1} \quad \left(\text{because } \left(y+\tfrac{1}{m}\right)^k y^{n-1-k} \leq (y+1)^k (y+1)^{n-1-k}\right)$$

By Archimedean property, we can choose m s.t. $m > \frac{n(y+1)^{n-1}}{x-y^n} \Leftrightarrow \frac{1}{m} < \frac{x-y^n}{n(y+1)^{n-1}}$

$= (y+1)^{n-1}$

then $\left(y+\frac{1}{m}\right)^n - y^n \leq x - y^n$ that is $\left(y+\frac{1}{m}\right)^n \leq x$.