Analysis I
UP - FAMNIT
Fall 2026 October 14-15, 2026

Tutorial 2

1. Show that $\mathbb{Q}$ is not well-ordered under its usual order.
2. (a) Prove that $\big||x| - |y|\big| \leq |x - y|$ for all $x, y \in \mathbb{R}$.
   (b) Determine all $x \in \mathbb{R}$ satisfying $|x - 2| + |x + 1| \leq 5$.
3. Prove that $\sqrt{2} + \sqrt{17}$ is irrational.
4. Prove that $\sqrt{3}$ is irrational. (*Left as an exercise*)
5. Prove that the square root of a positive natural number is either an integer or irrational.
6. Describe $3 - \sqrt{2}$ as a Dedekind cut of $\mathbb{Q}_{\geq 0}$ without explicitly referring to any irrational number in the definition of the cut.
7. Describe $\sqrt[3]{2}/5$ as a Dedekind cut of $\mathbb{Q}_{\geq 0}$ without explicitly referring to any irrational number in the definition of the cut. (*Left as an exercise*)
8. Let $\alpha \in \mathbb{R} \setminus \{0\}$ and $S \subseteq \mathbb{R}$ be nonempty. Define $\alpha S = \{\alpha x : x \in S\}$. Prove that αS is bounded if and only if S is bounded.
9. Decide, without a proof, if the following subsets of $\mathbb{R}$ are bounded from above/below. Give some bounds if that is the case. Find $\inf_{\mathbb{R}} A$ and $\sup_{\mathbb{R}} A$, or explain why either does not exist in $\mathbb{R}$. (*Parts (a), (b), (c), and (d) are left as exercises.*)

   (a) $\{0\} \cup \{x \ : \ x > 0\}$ (c) $\{x^3 \ : \ x \in \mathbb{Z}\}$ (e) $\{q \in \mathbb{Q} : q > 0\}$.
   (b) $\{x^2 \ : \ x < -2\}$ (d) $\{x^2 - x \ : \ x \geq 1\}$ (f) $\{1 - \frac{1}{n} : n \in \mathbb{N}\}$

10. Let $S = \left\{ \dfrac{2x + 5\cos(2x)}{x^2 + 1} : x \in \mathbb{R} \right\}$. Find an explicit lower and upper bound for S.

Some Solutions

5. Let $\sqrt{n} = a/b \in \mathbb{Q}$ and $a, b > 0$ with $gcd(a, b) = 1$. Then $a^2 = nb^2$. Assume $b > 1$. Then there exists a prime p such that $p \mid b$. Since

$$a^2 = nb^2,$$

we have $p \mid a^2$. As p is prime, this implies $p \mid a$. Thus p divides both a and b, contradicting $\gcd(a, b) = 1$. Therefore $b = 1$, and hence

$$\sqrt{n} = a \in \mathbb{N}.$$

So if $\sqrt{n}$ is rational, it must be an integer. Otherwise, it is irrational.

7. We want to represent $\dfrac{\sqrt[3]{2}}{5}$ as a Dedekind cut using only rational numbers. Since $\left(\frac{\sqrt[3]{2}}{5}\right)^3 = \frac{2}{125}$, we define

$$A = \left\{q \in \mathbb{Q}^{\geq 0} : 125q^3 < 2\right\}.$$

This definition does not explicitly refer to any irrational number. To verify that A is a Dedekind cut (in the nonnegative rationals):

(a) $A \neq \emptyset$ since $0 \in A$ and $A \neq \mathbb{Q}^{\geq 0}$ since $1 \notin A$.

(b) Contains all smaller elements: If $y \in A$ and $0 \leq x < y$, then $125x^3 < 125y^3 < 2$, so $x \in A$.

(c) No maximum element: For any $y \in A$, we have $125y^3 < 2$. Since the inequality is strict, we can choose a sufficiently small rational $\varepsilon > 0$ such that

$$125(y + \varepsilon)^3 < 2.$$

Thus $y + \varepsilon \in A$ and $y + \varepsilon > y$. Thus, we can always find an element of A greater than y. Therefore, A has no greatest element.

9.

$S \subseteq \mathbb{R}$	A lower bound	An upper bound	$\min S$	$\max S$	$\inf S$	$\sup S$
$\{0\} \cup \{x \ : \ x > 0\}$	0	—	0	—	0	—
$\{x^2 \ : \ x < -2\}$	0	—	—	—	4	—
$\{x^3 \ : \ x \in \mathbb{Z}\}$	—	—	—	—	—	—
$\{x^2 - x \ : \ x \geq 1\}$	0	—	0	—	0	—
$\{q \in \mathbb{Q} : q > 0\}$	0	—	—	—	0	—
$\{1 - \frac{1}{n} : n \in \mathbb{N}\}$	0	1	0	—	0	1