I-5.6. Existence of roots.

Theorem Let $x > 0$. Let $n \in \mathbb{N}$. There exists a unique $y > 0$ s.t. $y^n = x$.

Proof: Let us define $E = \{ t \in \mathbb{R} \mid t > 0 \text{ and } t^n < x \}$.

→ E is non-empty: as $\alpha = \frac{x}{1+x} \in E$. Indeed, $0 < \alpha < 1$

$\Leftrightarrow \frac{x}{1+x} > 0$ because $x > 0$ and $\alpha < 1$ because $\frac{x}{1+x} < \frac{1+x}{1+x} = 1$. Moreover as $1+x > 1$ and $x > 0$ we have $\alpha = \frac{x}{1+x} < x$. Now $\alpha^n \leq \alpha$ because $\alpha < 1$ (exercise: $n = 1$ true, valid for k: $\alpha^{k+1} = \alpha^k \cdot \alpha \leq \alpha \cdot \alpha \leq \alpha^2 \leq \alpha$)

Hence $\alpha^n \leq \alpha < x$ and thus $\alpha \in E$.

→ E is bounded above. Take $t_0 = 1+x$. If $t > 1+x$, then $t^n \geq t > 1+x > x$ so $t \notin E$. Contradiction, hence $1+x$ is an upper bound for E. (exercise for $t \geq 1$)

By the supremum property, there is a supremum $y = \sup E$. (to prove next)

2.1 Show $y^n = x$. Assume $y^n < x$. Then

$$\left(y + \frac{1}{m}\right)^n - y^n = \frac{1}{m} \cdot \left(\left(y+\frac{1}{m}\right)^{n-1} + \left(y+\frac{1}{m}\right)^{n-2} y + \dots + \left(y+\frac{1}{m}\right) y^{n-2} + y^{n-1} \right) \quad \text{(exercise)}$$

$$\leq \frac{1}{m} \cdot n \cdot (1+y)^{n-1} \quad \left(\text{because } \left(y+\frac{1}{m}\right)^k y^{n-1-k} \leq (y+1)^k (y+1)^{n-1-k} = (y+1)^{n-1}\right)$$

By Archimedean property, we can choose $m \in \mathbb{N}$ s.t. $m > \frac{n(y+1)^{n-1}}{x - y^n} \Leftrightarrow \frac{1}{m} < \frac{x-y^n}{n(y+1)^{n-1}}$

But $y - \frac{1}{n} > y$ which contradicts the claim that y is an upper bound for E.

Assume $y^n > x$. There exists $n \in \mathbb{N}$ s.t. $y - \frac{1}{n} > 0$.

Then $y^n - (y-\frac{1}{n})^n = \frac{1}{n}\left(y^{n-1} + y^{n-2}(y-\frac{1}{n}) + \dots + (y-\frac{1}{n})^{n-1}\right)$

$$\leq \frac{1}{n}\left(y^{n-1} + y^{n-1} + \dots + y^{n-1}\right)$$

$$= \frac{n}{n} y^{n-1}.$$

By Archimedean property we can choose

$n > \frac{n y^{n-1}}{y^n - x}$ (that is $\frac{1}{n} < \frac{y^n - x}{n y^{n-1}}$).

then $y^n - (y-\frac{1}{n})^n \leq \frac{n}{n} \cdot y^{n-1} < \frac{y^n - x}{n y^{n-1}} \cdot n \cdot y^{n-1} = y^n - x$

Hence $y^n - (y-\frac{1}{n})^n < y^n - x$, that is $x < (y-\frac{1}{n})^n$.

Now take $t \in E$. From $t^n < x$ and $x < (y-\frac{1}{n})^n$

we obtain that $t^n < (y-\frac{1}{n})^n$. As $t, y - \frac{1}{n} > 0$,

this implies that $t < y - \frac{1}{n}$. We use that for $a > b > 0$

and $n \in \mathbb{N}$ that $a^n > b^n > 0$ for every $n \in \mathbb{N}$. This is an easy exercise by

Hence $y - \frac{1}{n} < y$ is an upper bound for E. This contradicts the fact that $y = \sup E$.

~~This contradicts the fact that y~~

By trichotomy: $y^n = x$.

We write $y = x^{\frac{1}{n}}$ or $y = \sqrt[n]{x}$. When $n=2$, we often write

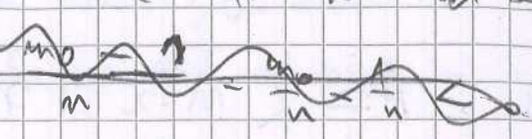
n-th root of x $\sqrt{x}$ for $\sqrt[2]{x}$.

Square root.

⚠ There might be more than 1 solution to $y^n = x$ if y is not restricted to be positive. Only two positive solution is called the n-th

we can choose m_0 to be the minimal element of S.

then $m_0 - 1 \leq nx < m_0$. Note that $q := \frac{m_0}{n}$ satisfies $x < q$. On the other hand



on the other hand $m_0 \leq nx + 1 < ny$, then $q = \frac{m_0}{n} <$

We have found a natural number q satisfying $x < q < y$.

I - Irrational numbers. (After next

We want to show that $\mathbb{Q} \subsetneq \mathbb{R}$.

Definition: We let a and $b \in \mathbb{Z}$. We say that $a \mid b$ there exists an integer k s.t. $k \cdot a = b$. We say that $n \in$ even if $2 \mid n$ and odd if $2 \nmid n$. (exercise: show n then n ($\nmid$ does not divide) for

Lemma: If n^2 is even then n is even.

Proof: Assume (by contrapositive) n is odd. then $n = 2k+1$. We have $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2 \cdot (2k^2 + 2k) + 1$ odd, a contradiction. Hence n is even.

Theorem: Let $x := \sqrt{2}$. One has that

Proof: Assume by the contrary that $x \in \mathbb{Q}$. Then $x = \frac{m_0}{n_0}$ with $n_0 \in \mathbb{N}$ and $n_0 \neq 0$. We have that $2 = x^2 = \frac{m_0^2}{n_0^2}$ that is $m_0^2 = 2n_0^2$. Let $S = \{n \in \mathbb{N} \mid \text{there exists } m \in \mathbb{Z} \text{ s.t. } m^2 = 2n^2\}$

Note $n_0 \in S$, hence S is non-empty. $\Rightarrow$ by Well ordering of $\mathbb{N}$, S has a minimal element, call it n_1. Then for some $m_1 \in \mathbb{Z}$ $m_1^2 = 2n_1^2$. $2 \mid m_1^2 \Rightarrow m_1$ even. $\Rightarrow$ $m_1 = 2k$ $4k^2 = 2n_1^2$ $2k^2 = n_1^2$. but $k^2 < 2k^2 < n_1^2 \Rightarrow k < n_1$, of $k \in S$, a contradict

I-5-8 Decimal expansion.

Theorem (Decimal expansion of real numbers). For every real number $x \geq 0$, there exists an integer $a_0 \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and some integers $a_1, a_2, a_3, \ldots$ with $a_i \in \{0, 1, \ldots, 9\}$ such that for each $k \geq 1$

$$x = \sup \left(a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \cdots + \frac{a_k}{10^k} \right)$$

We write $x = a_0, a_1 a_2 \ldots$

Remark: We can choose any other basis $b \geq 2$.

Proof: Step 1: as before. Let $S := \{n \mid n > x\}$ this is a non-empty subset of $\mathbb{N}$, by the Archimedean property. Let $n_0 := \min(S)$ (minimal element of $\mathbb{N}$). Then we define $a_0 := n_0 - 1 \in \mathbb{N}_0$. We have that $a_0 \leq x < a_0 + 1$.

Step 2: We Assume we have defined $a_0, a_1 \ldots a_{k-1}$ for some $k \in \mathbb{N}$. We denote $E_{k-1} := a_0 + \sum_{i=1}^{k-1} \frac{a_i}{10^i}$.

Then $E_{k-1} \leq x < E_{k-1} + \frac{1}{10^{k-1}}$.

Let a_k be the largest integer in $\{0, 1, \ldots, 9\}$ s.t. $E_{k-1} + \frac{a_k}{10^k} \leq x$.

Then $E_k := E_{k-1} + \frac{a_k}{10^k}$ satisfies $E_k \leq x < E_k + \frac{1}{10^k}$.

Step 3: By construction $E_k \leq x$ for every $k \in \mathbb{N}_0$. So x is an upper bound for the set $A = \{E_k \mid k \in \mathbb{N}_0\}$.

Assume $x \neq y$, then

Let $y := \sup A$. Since $x > y$. By the Archimedean property there exists k s.t. $\frac{1}{10^k} < x - y$. (we have $10^k \geq k \Leftrightarrow$ shown by induction)

Then $x < E_k + \frac{1}{10^k}$ gives $E_k > x - \frac{1}{10^k} > x - (x - y) = y$.

We obtain $E_k > y$ and y is not an upper bound for E_k, a contradiction.

Hence $x = y = \sup(A) = \sup\{E_k\}$.

Remark: We omit 0 at the end

e.g. write $1 = 1,0\ldots$ One has that $0 = 0,0\ldots$

We write negative numbers by putting $-$ in front e.g. $-1,234\ldots$

Example: $\sqrt{2} = 1{,}4142\ldots$

$\Leftrightarrow 1 < 2$

$a_0 = 1 \qquad 1 < \sqrt{2} < 2$

Lemma

1. $1 \leq \sqrt{2}$ (Indeed for $a, b \geq 0$ we have that $a \leq b \Leftrightarrow$

$a \leq b \Rightarrow a^2 \leq ab \leq b^2$

$a \geq 0 \quad b \geq 0$

$a^2 \leq b^2 \Leftrightarrow b^2 - a^2 \geq 0 \quad (b-a)$

$a + b > 0 \Rightarrow b - a \geq 0$

Because

$1 \leq 2$.

2. $\sqrt{2} < 2 \Leftrightarrow 2 < 4$ ok.

Thus $a_0 \geq 1$.

2. [redacted] < 2

a_1 is the largest s.t. $\left(1 + \frac{a_1}{10}\right)^2 < \sqrt{2} \Leftrightarrow \left(1 + \frac{a_1}{10}\right)^2 < 2$

$a_1 = 4 \qquad (1{,}4)^2 = \left(1 + \frac{4}{10}\right)^2 = 1 + \frac{8}{10} + \frac{16}{100} = 1 + \frac{80}{100} + \frac{16}{100}$

$= 1{,}96$

$a_1 = 5 \qquad (1{,}5)^2 = 2{,}25 > 2.$

$\Rightarrow a_1 \leq 4$

$\Rightarrow a_1 = 4$

$a_2 = 1 \ldots$

10^{13} digits currently known

I-5-9 Absolute value

Definition: Given $x \in \mathbb{R}$, we define

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Theorem: Given $a, b \in \mathbb{R}$, we have that:

a) $|a+b| \leq |a| + |b|$

b) $|a+b| \geq |a| - |b|$

c) $|ab| = |a| \cdot |b|$

d) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ if $b \neq 0$

Let $\varepsilon > 0$, $u \in \mathbb{R}$

e) $|x - u| < \varepsilon \Leftrightarrow x \in (u - \varepsilon, u + \varepsilon)$

ε-neighbourhood of u