Analysis I
UP - FAMNIT
Fall 2026 October 22, 2026

Tutorial 3

1. Let $a > 0$ and $n \in \mathbb{N}$, and set $S = \{t \in \mathbb{R} : t > 0,\ t^n < a\}$. Show that S is nonempty and bounded above.

2. Let $x \notin \mathbb{Q}$ and $r \in \mathbb{Q}$.

   (a) Show that $x + r$ is irrational.

   (b) If $r \neq 0$, show that rx is irrational.

   (c) Give two irrational numbers whose product is rational.

3. Let $x = 7/8$.

   (a) Find its decimal expansion.

   (b) Write $x = a_0 + \sum_{k=1}^{\infty} a_k 10^{-k}$ with $a_0 \in \mathbb{N}_0$ and $a_k \in \{0, \ldots, 9\}$.

   (c) Prove that $0.999\ldots = 1$.

4. Without a calculator, find the first two digits after the decimal point of $\sqrt{2}$ by comparing squares of consecutive decimal approximations.

5. Let $a < b$.

   (a) Prove $x \in (a, b) \iff \left|x - \frac{a+b}{2}\right| < \frac{b-a}{2}$.

   (b) Express $(-3, 7)$ by one absolute-value inequality.

   (c) Solve $|x - 1| \geq |x + 3|$.

Some Solutions

3. (a) $7/8 = 0.875$.
   (b) $a_0 = 0$, $a_1 = 8$, $a_2 = 7$, $a_3 = 5$, and $a_i = 0$ for $i \geq 4$.
   (c) The partial sums are

$$s_n = \sum_{i=1}^{n} 9 \cdot 10^{-i} = 1 - 10^{-n}.$$

   $0.999\ldots = \lim_{n\to\infty} s_n = 1$ since $10^{-n} \to 0$.

   Let

$$A = \left\{ 1 - \frac{1}{10^n} : n \in \mathbb{N} \right\}.$$

   By the definition of decimal expansion,

$$0.999\ldots = \sup A.$$

   Clearly, 1 is an upper bound of A.

   Let $\varepsilon > 0$. By the Archimedean property, there exists $n \in \mathbb{N}$ such that

$$10^n > \frac{1}{\varepsilon}.$$

   Hence

$$1 - \varepsilon < 1 - \frac{1}{10^n} \in A.$$

   Thus no number smaller than 1 is an upper bound of A, and therefore

$$0.999\ldots = \sup A = 1.$$

4. Since $1.4^2 = 1.96 < 2 < 2.25 = 1.5^2$, the first decimal digit is 4. Since $1.41^2 = 1.9881 < 2 < 2.0164 = 1.42^2$, the second is 1.

5. (a) Suppose that $x \in (a, b)$. Then $a < x < b$. Subtracting $\frac{a+b}{2}$ from each part of the inequality, we obtain

$$\frac{a-b}{2} < x - \frac{a+b}{2} < \frac{b-a}{2}.$$

   Since $\frac{a-b}{2} = -\frac{b-a}{2}$, this is equivalent to

$$\left| x - \frac{a+b}{2} \right| < \frac{b-a}{2}.$$

   (b) The midpoint is 2 and the radius is 5, so $|x - 2| < 5$.
   (c) Both sides are nonnegative; squaring gives $(x-1)^2 \geq (x+3)^2$, or $-8x \geq 8$. Thus $x \leq -1$.