MATHEMATICAL METHODS

LECTURE 5

1. Number sets

1.1. **Real numbers.** The fact that $\mathbb{R}$ satisfies axiom (D) implies the following widely used properties of real numbers:

(1) The archimedean property: If $(x, y) \in \mathbb{R}^2$ and $x > 0$, then there exists $n \in \mathbb{N}$ such that $nx > y$;
(2) $\mathbb{Q}$ is *dense* in $\mathbb{R}$: If $(x, y) \in \mathbb{R}^2$ and $x < y$, then there exists $q \in \mathbb{Q}$ such that $x < q < y$;
(3) The existence of nth roots of positive real numbers: Let x be a positive real number and $n \in \mathbb{N}$. Then there exists a unique positive real number y such that $y^n = x$. The number y is written as $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$.

1.2. **Intervals.** Let $a \in \mathbb{R}$ and $b \in \mathbb{R}$ such that $a < b$. Then we define the following subsets of $\mathbb{R}$:

(1) The *open interval* $(a, b) = \{x \in \mathbb{R} \colon a < x < b\}$;
(2) The *closed interval* $[a, b] = \{x \in \mathbb{R} \colon a \leq x \leq b\}$;
(3) The *half-open* (or *half-closed*) intervals $[a, b) = \{x \in \mathbb{R} \colon a \leq x < b\}$ and $(a, b] = \{x \in \mathbb{R} \colon a < x \leq b\}$.

We usually denote an arbitrary interval (open, closed or half-open) by $\mathcal{I}$. Let $x \in \mathbb{R}$ and $\varepsilon > 0$. Then the interval $(x - \varepsilon, x + \varepsilon)$ is called the *ε-neighbourhood of x*.

1.3. **Decimal notation.** If $x \in \mathbb{R}$, then there exist $a_0 \in \mathbb{Z}$ and $a_1 \in \{0, 1, \ldots, 9\}, a_2 \in \{0, 1, \ldots, 9\}, \ldots$ such that

$$x = a_0 + a_1 \cdot 10^{-1} + a_2 \cdot 10^{-2} + \cdots + a_k \cdot 10^{-k} + \cdots .$$

This is a *decimal notation* of x, in short as $x = a_0.a_1a_2 \ldots a_k \ldots$. For example, $1/2 = 0.5$, $1/3 = 0.3333\ldots = 0.\bar{3}$, $1/7 = 0.\overline{142857}$, $\sqrt{2} = 1.41421\ldots$. We know that $x \in \mathbb{Q}$ if and only if its decimal notation is *finite* ($\exists K \in \mathbb{N}.\ \forall k \geq K.\ a_k = 0$) or *periodic* ($\exists K \in \mathbb{N}.\ \exists d \in \mathbb{N}.\ \forall k \geq K.\ a_k = a_{k+d}$).

1.4. **Real functions.** Let $D \subseteq \mathbb{R}$. A map $f \colon D \to \mathbb{R}$ is called a *real function*. The existence of $0 \in \mathbb{R}$ implies the following definition: $x_0 \in \mathbb{R}$ is called a *zero* of f if $f(x_0) = 0$. That $\mathbb{R}$ has an order implies the following definitions:

(1) $f \colon \mathcal{I} \to \mathbb{R}$ is an *increasing function* on $\mathcal{I}$ if for every $(x_1, x_2) \in \mathcal{I} \times \mathcal{I}$, $x_1 \leq x_2$ implies $f(x_1) \leq f(x_2)$;
(2) $f \colon \mathcal{I} \to \mathbb{R}$ is a *strictly increasing function* on $\mathcal{I}$ if for every $(x_1, x_2) \in \mathcal{I} \times \mathcal{I}$, $x_1 < x_2$ implies $f(x_1) < f(x_2)$;
(3) $f \colon \mathcal{I} \to \mathbb{R}$ is a *decreasing function* on $\mathcal{I}$ if for every $(x_1, x_2) \in \mathcal{I} \times \mathcal{I}$, $x_1 \leq x_2$ implies $f(x_1) \geq f(x_2)$;
(4) $f \colon \mathcal{I} \to \mathbb{R}$ is a *strictly decreasing function* on $\mathcal{I}$ if for every $(x_1, x_2) \in \mathcal{I} \times \mathcal{I}$, $x_1 < x_2$ implies $f(x_1) > f(x_2)$;

A *monotonic* function $f \colon \mathcal{I} \to \mathbb{R}$ is a function that increases or decreases on $\mathcal{I}$. A *periodic* function $f \colon \mathbb{R} \to \mathbb{R}$ with a *period* $P > 0$ is such that $f(x + P) = f(x)$ for all $x \in \mathbb{R}$.

Example 1. Here we list some frequently used real functions:

(1) *Polynomials*: $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_n$ are real numbers. If $a_n \neq 0$, then it is said that a polynomial has *degree* n. Polynomials are defined on $\mathbb{R}$. A polynomial of degree n has at most n zeros.
(2) *Rational functions*: $p(x)/q(x)$, where $p(x)$ and $q(x) \not\equiv 0$ are polynomials such that $p(x)$ and $q(x)$ do not have a common zero. Domain: $\mathbb{R} \setminus \{x \in \mathbb{R} \colon q(x) = 0\}$.

Date: 27. 10. 2026.

(3) *Trigonometric functions*: sine $\sin(x)$, cosine $\cos(x)$, tangent $\tan(x)$, cotangent $\cot(x)$. Sine and cosine are periodic with a period 2π, while tangent and cotangent have period π. Domain of sine and cosine is $\mathbb{R}$, while their range is $[-1, 1]$. Domain of tangent is $\mathbb{R} \setminus \{\pi/2 + k\pi \colon k \in \mathbb{Z}\}$ and domain of cotangent is $\mathbb{R} \setminus \{k\pi \colon k \in \mathbb{Z}\}$. Range of both functions is $\mathbb{R}$.

(4) The *exponential function* a^x for $a \in \mathbb{R}^+$, and the *logarithmic function* $\log_a x$ for $a \in \mathbb{R}^+ \setminus \{1\}$.
   (a) a^x: Domain is $\mathbb{R}$ and range is $\mathbb{R}^+$. We have the following: if $a > 1$, then a^x is strictly increasing; if $a < 1$, then a^x is strictly decreasing; if $a = 1$, then $a^x \equiv 1$. Arguably the most important function in mathematical analysis is the exponential function with $a = e$, where $e \approx 2.71828$ is the *Euler constant*. **Important: In mathematical and scientific literature, e^x is sometimes denoted as $\exp(x)$.**
   (b) $\log_a x$: Domain is $\mathbb{R}^+$ and range is $\mathbb{R}$. It is the inverse function of a^x, i.e., $y = a^x$ if and only if $x = \log_a x$. If $a > 1$, then $\log_a x$ is strictly increasing, while for $a \in (0, 1)$ it is strictly decreasing. It has only one zero $x = 1$. Change of basis: if $b \in \mathbb{R}^+ \setminus \{1\}$, then $\log_a x = (\log_b x) / (\log_b a)$. The *natural logarithm*: $\ln x = \log_e x$. **Important: In mathematical and scientific literature, $\ln$ is usually denoted simply as $\log$.**

1.5. Some applications of the exponential function.

1.5.1. *Natural growth.* The law of natural growth is given as a function

$$y(t) = y_0 e^{kt}, \tag{1}$$

where:

(1) $y(t)$ models the growth of something, e.g., a population, in time t;
(2) y_0 is the initial ($t = 0$) size;
(3) $k > 0$ is the growth coefficient, constant for a particular model.

1.5.2. *Logistic law.* The obvious problem with (1) is that $y(t)$ growths to infinity as t growths to infinity. So, for large t this does not seem to be a realistic model of a growth of something from real world. One simple way to correct (1) by assuming that $y(t)$ approaches some finite value when t approaches infinity, is the following:

$$y(t) = \frac{K}{1 + \left(\frac{K}{y_0} - 1\right) e^{-kt}}.$$

Here, $K > y_0$ is called the *carrying capacity* (of the environment). All other parameters are as the above.

1.5.3. *Natural decay.* This model is the same as that given by (1), but now we have $k < 0$. The most common example is radioactivity. Here, $t_{1/2}$ is the parameter called *half-life: the time required for a quantity of a substance to reduce to half of its initial value.* This means that if N_0 is the initial quantity of a substance, then after $t_{1/2}$ there will be $N_0/2$ quantity of a substance, and after $2t_{1/2}$ there will be $N_0/4$ quantity of a substance etc. Therefore,

$$N(t) = N_0 \cdot 2^{-t/t_{1/2}},$$

if $N(t)$ is the quantity of a substance after time t. Transforming the above equation in the form of (1), we obtain

$$N(t) = N_0 e^{-kt}, \quad \text{where} \quad k = \frac{\ln 2}{t_{1/2}}.$$

Example 2. Radiocarbon dating is a method for determining the age of an object containing organic material by using the properties of radiocarbon ^{14}C, a radioactive isotope of carbon. We know that $t_{1/2} \approx 5700$ years, giving $k \approx 0.0001216$ per years. The most basic problem would be the following: Assume that the quantity of ^{14}C in some archeological object at the moment of creation was N_0, while the current quantity is N. Determine the age T[yr] of the object. We have $N = N_0 e^{-kT}$, and thus $T = (1/k) \ln(N_0/N)$, where $k = 0.0001216\ \text{yr}^{-1}$.