Analysis I
UP - FAMNIT
Fall 2026 October 28-29, 2026

Tutorial 4

1. Prove that for $z = a + bi \neq 0$,
$$\frac{1}{z} = \frac{a - bi}{a^2 + b^2}.$$
Use your result to compute $(1 + 2i)^{-1}$.

2. Find all complex numbers z satisfying
$$|z - 1| = |z - i|, \qquad |z| = \sqrt{8}.$$
(*Left as an exercise*)

3. Let $a_1 = 1$ and $a_{n+1} = 2a_n + 1$ for $n \geq 1$.
   (a) Find the first five terms.
   (b) Guess a formula for a_n and prove it by induction.
   (c) Is the sequence monotone? Is it bounded?

4. Let $a_1 = 1$ and
$$a_{n+1} = \frac{a_n + 2}{2} \qquad (n \geq 1).$$
   (a) Prove by induction that $1 \leq a_n < 2$ for every $n \geq 1$.
   (b) Prove that (a_n) is strictly increasing.
   (c) Find an explicit formula for a_n and verify it by induction.

5. Consider $a_n = (-1)^n \left(1 + \frac{1}{n}\right)$.
   (a) Write the first five terms of (a_n).
   (b) Find explicit formulas for $(a_{2k})_{k\geq 1}$ and $(a_{2k-1})_{k\geq 1}$.
   (c) Is the original sequence bounded? Is it monotone?

   (*Left as an exercise*)

Some Solutions

2. Let $z = x + iy$, where $x, y \in \mathbb{R}$. Then

$$|z - 1| = |z - i|$$

implies

$$(x-1)^2 + y^2 = x^2 + (y-1)^2.$$

Simplifying, we obtain

$$x = y.$$

Moreover, since $|z| = \sqrt{8}$,

$$x^2 + y^2 = 8.$$

Substituting $y = x$, we get

$$2x^2 = 8 \implies x = \pm 2.$$

Since $x = y$, the solutions are

$$z = 2 + 2i \quad \text{or} \quad z = -2 - 2i.$$

5. (a) The first five terms are $a_1 = -2,\ a_2 = \frac{3}{2},\ a_3 = -\frac{4}{3},\ a_4 = \frac{5}{4},\ a_5 = -\frac{6}{5}$.

(b) For the even-indexed subsequence, let $n = 2k$. Then

$$a_{2k} = (-1)^{2k}\left(1 + \frac{1}{2k}\right) = 1 + \frac{1}{2k}, \qquad k \geq 1.$$

For the odd-indexed subsequence, let $n = 2k - 1$. Then

$$a_{2k-1} = (-1)^{2k-1}\left(1 + \frac{1}{2k-1}\right) = -1 - \frac{1}{2k-1}, \qquad k \geq 1.$$

(c) For every $n \geq 1$, we have

$$1 < 1 + \frac{1}{n} \leq 2.$$

Hence,

$$|a_n| = 1 + \frac{1}{n} \leq 2 \quad \implies \quad -2 \leq a_n \leq 2$$

Therefore, the sequence is bounded. However, this does not mean that the sequence attains both endpoints. It only shows that the sequence is bounded. In fact, since the even-indexed terms satisfy

$$a_{2k} = 1 + \frac{1}{2k} \leq \frac{3}{2},$$

we obtain the sharper bounds $-2 \leq a_n \leq \frac{3}{2}$.

The sequence is neither increasing nor decreasing. Indeed,

$$a_1 = -2 < a_2 = \frac{3}{2},$$

but

$$a_2 = \frac{3}{2} > a_3 = -\frac{4}{3}.$$

Thus, the sequence is **not monotone**.

Notice also that the even-indexed subsequence (a_{2k}) is strictly decreasing, whereas the odd-indexed subsequence (a_{2k-1}) is strictly increasing.