Example: $\sqrt{2} = 1{,}4142\ldots$

$\Leftrightarrow 1 < 2$

$a_0 = 1 \qquad 1 < \sqrt{2} < 2$

Lemma:

1. $1 \leq \sqrt{2}$ (Indeed) For $a, b > 0$ we have that $a \leq b \Leftrightarrow$

$a \leq b \Rightarrow a^2 \leq ab \leq b^2$

$a>0 \quad b>0$

$a^2 \leq b^2 \Leftrightarrow b^2 - a^2 \geq 0 \quad (b-a)$

$a+b>0 \Rightarrow b-a \geq 0$

Because $1 \leq 2$.

2. $\sqrt{2} < 2 \Leftrightarrow 2 < 4$ ok.

Thus: $a_0 = 1$.

2. [redacted] < 2

a_1 is the largest s.t. $\left(1 + \frac{a_1}{10}\right)^2 < \sqrt{2} \Leftrightarrow \left(1 + \frac{a_1}{10}\right)^2 < 2$

$a_1 = 4 \qquad (1{,}4)^2 = \left(1 + \frac{4}{10}\right)^2 = 1 + \frac{8}{10} + \frac{16}{100} = 1 + \frac{80}{100} +$

$= 1{,}96$

$a_1 = 5 \qquad (1{,}5)^2 = 2{,}25 > 2.$

$\Rightarrow a_1 = 4$

10^{13} digits currently known

3. $a_2 = 1 \ldots$

I-5-9 Absolute value

Definition: Given $x \in \mathbb{R}$, we define

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Theorem: Given $a, b \in \mathbb{R}$, we have that:

a) $|a+b| \leq |a| + |b|$

b) $|a+b| \geq |a| - |b|$

c) $|ab| = |a| \cdot |b|$

d) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ if $b \neq 0$

Let $\varepsilon > 0$, $x \in$

e) $|x - $ [redacted] $\Leftrightarrow x \in$

(ε-neighbourhood of a)

Proof: a) We have that $a \le |a|$ (if $a \ge 0$ then $|a| = a$; if $a < 0$ then $a < 0 < -a = |a|$) and that $-a \le |a|$ (if $a \ge 0$, then $-a < 0 < a = |a|$; if $a < 0$ then $-a = |a|$).

If $a+b \ge 0$ we obtain that

$$|a+b| = a+b \le |a| + |b|$$

If $a+b < 0$, then $|a+b| = -(a+b) = -a + (-b) \le |a| + |b|$

Equality holds if $a, b \ge 0$ simultaneously or if $a, b \le 0$ simultaneously.

b) We have that $|a| = |(a+b) + (-b)| \le |a+b| + |-b| = |a+b| + |b|$

and hence $|a| - |b| \le |a+b|$. One has that $|x| = |-x|$ immediately from definition

c) If $a, b \ge 0$ then $|ab| = ab$ implies $\Rightarrow |ab| = |a| \cdot |b|$

$|a||b| = ab$

If $a, b \le 0$ then $ab \ge 0$, thus $|ab| = ab$ $\Rightarrow |ab| = |a||b|$

$|a||b| = (-a)(-b)$

$= ab$

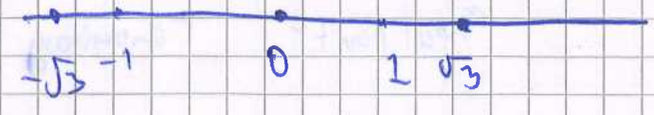
If $a \ge 0 \ge b$ then $ab \le 0$, $|ab| = -ab$ $\Rightarrow |ab| = |a| \cdot |b|$

$|a||b| = a \cdot (-b)$

The case $b \ge 0 \ge a$ follows by symmetry

d) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ $\rightsquigarrow$ ~~absurdo~~. Et cetera.

Exercise: Show (by induction) that $|a_1 + \dots + a_n| \le |a_1| + \dots + |a_n|$.



$|x|$ denotes "distance" to 0.

I-C Complex numbers. $\mathbb{C}$

I-C-1 Cartesian product of sets

Def An ordered pair (a,b) consists of a first entry a and second entry b

We The defining property is: $(a,b) = (c,d) \Leftrightarrow a = c$ and $b = d$

Remark: (a,b) can be defined formally as $\{a, \{a,b\}\}$

Definition: The Cartesian product of sets A and B is the set of ordered pairs:

$$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$$

⚠ $\{1,2\} = \{2,1\}$

$(1,2) \neq (2,1)$

I-6-2

Definition: We define complex numbers $\mathbb{C}$ as $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$ with addition $(a,b) + (c,d) = (a+c, b+d)$ and multiplication

$$(a,b) \cdot (c,d) = (ac-bd, ad+bc)$$

The real number $a \in \mathbb{R}$ is a complex number $(a,0)$.

For pair (a,b), we write: $a+bi$. (We write only a for $a+0i$)

Properties: • Commutativity / associativity of addition and multiplication

• Distributivity • 1 neutral for multiplication, 0 for addition

• Existence of mult. inverses if $a+bi \neq 0$, $\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$

It is easy to show that

We have seen that if $a \in \mathbb{R}$, then $a^2 \geq 0$. This means that the equation $z^2 = -1$ does not have solutions in $\mathbb{R}$.

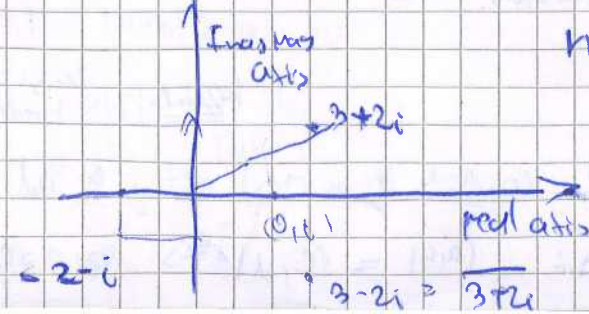
However in $\mathbb{C}$, we have that:

$$i \cdot i = (0,1) \cdot (0,1) = (-1,0) = -1 \quad !$$

Conjugation: If $z \in \mathbb{C}$, $z = a+bi$ $a,b \in \mathbb{R}$, then $\bar{z} = a-bi$. We write $\underset{\in \mathbb{R}}{Re(z)} = a$, $\underset{\in \mathbb{R}}{Im(z)} = b$

"real part" "imaginary part"

An geometric representation:



Modulus $|z| = \sqrt{a^2+b^2}$ = "distance to the origin"

$|z| = |zw|$

$\Leftrightarrow (ac-bd) + (ad+bc)i$

$|zw|^2 = |a(c-ba)^2 + |ad+bc|^2 = a^2c^2 + b^2d^2 + 2abcd + a^2d^2 + b^2c^2 - 2abcd$

$= a^2+b^2 \quad |c^2+d^2| = |z|^2 |w|^2$ $|z|, |w| \geq 0$

One has $|\mathrm{Re}(z)| \le |z|$, $|\mathrm{Im}(z)| \le |z|$, $|z|^2 = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2 \ge |\mathrm{Re}(z)|$

Multiply by i rotating counterclockwise by $90°$:

$i \cdot (a+bi) = ai + bi^2 = ai + b\cdot(-1) = -b + ai$.

Multiplying by -1 is symmetry w.r.t. origin

Conjugation = symmetry w.r.t. the real axis.

Ex.:

ε-neighbourhood of $\in \mathbb{C}$: $|z - x| < \varepsilon$



Proposition For $z, w \in \mathbb{C}$:

1) $\overline{\bar{z}} = z$ 2) $\overline{z+w} = \bar{z} + \bar{w}$ 3) $\overline{zw} = \bar{z}\,\bar{w}$

4) $z + \bar{z} = 2\,\mathrm{Re}\,z$ 5) $z - \bar{z} = 2i\,\mathrm{Im}(z)$ 6) $z\bar{z} = |z|^2$

7) $|z+w| \le |z| + |w|$ $\le |z|^2 + |w|^2 + 2|z\bar{w}| = (|z|+|w|)^2$

$\triangleright$ $(z+w)\cdot\overline{(z+w)} = z\bar{z} + w\bar{z} + \bar{w}z + w\bar{w} = |z|^2 + |w|^2 + 2\mathrm{Re}(z\bar{w})$

$|z+w|^2 =$ [illegible]

$\triangle$ There is no order relation! More precisely, it is impossible to define an order on $\mathbb{C}$, satisfying standard axioms 1) [illegible]

2) [illegible]

3. If $z < w$, $z+u < w+u$ [illegible] 4.) $z > 0$ $w > 0$ $\Rightarrow zw > 0$.

Consider i: we must have $i > 0$ or $i < 0$. If $i > 0$ then $-1 = i^2 > 0$. $\lightning$

If $i < 0$ then $0 < -i$ and $i^2 = (-1)^2 > 0$. Now ch

II Sequences

II-1 Sequences of real numbers.

A, B sets

Def.: A function $f: A \to B$ is a rule that assigns to each element $x \in A$ exactly one element $f(x)$ in B.

We call A domain and B codomain of f.

Example: $A = \{1,2,3\}$ $\quad B = \{4,5,6\}$

$f(1) = 4$
$f(2) = 4$ a function
$f(3) = 5$

$f(1) = 4$
$f(1) = 5$
$f(2) = 4$
$f(3) = 6$
} not a function!

Definition: A sequence of real numbers is a function $a: \mathbb{N} \to \mathbb{R}$. We write a_n instead of $a(n)$ and denote the sequence by $(a_n)_{n \in \mathbb{N}}$. The number a_n is called n-th term.

(also write $(a_n)_{n \geq 1}$ or (a_n))

A sequence can be written as an ordered list: $a_1, a_2, a_3, \ldots$

1 $(a_n)_{n \in \mathbb{N}}$ $a_n = n$

Examples: 1) The sequence $(a_n)_{n \in \mathbb{N}}$ given by $a_n = \frac{1}{n}$ has terms $1, \frac{1}{2}, \frac{1}{3}, \ldots$

$(a_n)_{n \in \mathbb{N}}$

2) The sequence $a_n = (-1)^n$ has terms

$-1, 1, -1, 1$

$(-1)^1$ $(-1)^2$

3) For fixed $c \in \mathbb{R}$, the sequence $a_n = c$ has terms

$c, c, c, c \ldots$ it is called a constant sequence.

The written writers: $a_n = (-1)^n$ $\quad b_n = (-1)^{n+1}$ are not the same!

$-1, 1, -1, \ldots$ $\quad 1, -1, 1, -1, \ldots$

Remark: Sequences can be defined recursively:

$a_1 = 1$
$a_{n+1} = 2a_n + 1$ for $n \in \mathbb{N}$

$1, 3, 7, 15, \ldots$

Def A sequence $(a_n)_{n\in\mathbb{N}}$ is: bounded above if $\exists M\in\mathbb{R}$ (one bound must work for all members) such that $a_n \le M$ for every $n\in\mathbb{N}$.

: bounded below if there exists $m\in\mathbb{R}$, such that $m\le a_n$ for every $n\in\mathbb{N}$

: bounded, if it is bounded both above and below.
(it is unbounded if it is not bounded) (by Archimedean property)

Examples: 1) $a_n = n$ is not bounded above but bounded below

2) $a_n = \frac{1}{n}$ is bounded $0 < a_n \le 1$ $\forall n\in\mathbb{N}$

3) $a_n = (-1)^n$ is bounded $-1 \le a_n \le 1$.

~~Given a sequence we say~~

Def A sequence (a_n) is called: ① increasing (also non decreasing) if $\forall n\in\mathbb{N}$: $a_n \le a_{n+1}$

② strictly increasing if $\forall n\in\mathbb{N}$: $a_n < a_{n+1}$

③ decreasing (also non-increasing) if $\forall n\in\mathbb{N}$ $a_n \ge a_{n+1}$

④ strictly decreasing if $\forall n\in\mathbb{N}$ $a_n > a_{n+1}$.

⑤ monotone if it is increasing or decreasing.

Examples: $a_n = \frac{1}{n}$ is strictly decreasing, decreasing and monotone.

$a_n = n$ is strictly increasing, increasing and monotone.

$a_n = 2$ is increasing, decreasing and monotone.

$a_n = (-1)^n$ is not monotone (hence not increasing / decreasing / strictly ...)
$a_n = (-1)^n$ not bounded above not bounded below!

Def: A subsequence of $(a_n)_{n\in\mathbb{N}}$ is a sequence $(a_{n_k})_{k\in\mathbb{N}}$, where / strictly decreasing
$(n_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers (that is $n_k\in\mathbb{N}$)

Ex. 1) $a_n = (-1)^n$ $n_k = 2k$ 2) $a_n = (-1)^n$ $n_k = 2k+1$

$\Rightarrow a_{n_k} = 1$ $\Rightarrow a_n = -1$.

Remark: the index n_k in a subsequence satisfies $n_k \ge k$. (by induction)

Proposition 1) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers then $(a_n)_{n\in\mathbb{N}}$ is bounded $\Leftrightarrow$ $\exists\, C>0$ such that $\forall n\in\mathbb{N}\quad |a_n|\leq C$

2) If $(a_n)_n$ and $(b_n)_n$ are bounded above, then the sequence $\to$ $(a_n+b_n)_n$ is bounded above.

n-th term is the number a_n+b_n.

Example: $a_n = n$ (1,2,3,...), $b_n = 2n$ (2,4,6,...) $\to$ $(a_n+b_n) = 3,6,9,\dots$

$a_n = (-1)^n$, $b_n = n$ $\Rightarrow$ $a_n+b_n = n+(-1)^n$

$-1,1,-1,1,-1,\dots$ $\quad 1,2,3,\dots$ $\quad 0,3,2,5,4,\dots$

~~3) if (a_n)~~ Same for bounded below / bounded.

3) if $(a_n)_{n\in\mathbb{N}}$ is bounded above then $(-a_n)_n$ is bounded below

~~and~~

if $(a_n)_n$ is bounded below, then $(-a_n)_n$ is bounded above.

if $(a_n)_n$ is bounded, then $(-a_n)_n$ is bounded.

$a_n = n \qquad -a_n = -n$

4) If $(a_n)_n$ is a sequence of real numbers, and $(n_k)_{k\in\mathbb{N}}$ a sequence of natural numbers, then the subsequence $(a_{n_k})_{k\in\mathbb{N}}$ is bounded above/below/bounded

which is bounded above/below/bounded

5) If $(a_n)_n$ is a sequence of real numbers, ~~and~~ $m\in\mathbb{N}$ and ~~$n\in\mathbb{N}$~~ by the ~~sequence~~ subsequence $(b_n)_n = (a_{(n+m+1)})_n$ is called a tail of (a_n)

~~If~~ One has that $(a_n)_n$ is bounded above/below/bounded

$a_1 a_2 \dots a_N a_{N+1} \dots$ $\Leftrightarrow$ $(b_n)_n$ is bounded above/below/bounded.

$b_1\ b_2\ \dots$

1) $\Rightarrow$

Proof. If $(a_n)_n$ is bounded $\exists\, m, M \in \mathbb{R}$ s.t. $m \le a_n \le M \ \forall n$. Set $C := \max(|M|, |m|)$.

then $-C \le m \le a_n \le M \le C$

In other words $|a_n| \le C$.

$\Leftarrow$ For the other direction note that $|a_n| \le C$ implies $-C \le a_n \le C$. Set $m := -C$ and $M := C$.

We have that $m \le a_n \le M$ $\forall n$, that is (a_n) is bounded.

max := sup $\{m, M\}$ must & sup are the same thing for finite sets

2) bounded above $\quad a_n \le M_1 \ \forall n$

$b_n \le M_2 \ \forall n$

The sequence $(a_n + b_n)_n$ satisfies $a_n + b_n \le M_1 + M_2$ $\forall n$, hence is bounded above. However: do the other two claims!

3) If a_n is bounded above: $\exists M$ s.t. $a_n \le M$ $\forall n$.

then $a_n - M \le 0$ $\forall n$, hence, $-M \le -a_n$ $\forall n$

Hence $(-a_n)_{n\in\mathbb{N}}$ is bounded below. However: do the other claim

4) If $(a_n)_n$ is bounded above $\exists M$ s.t. $a_n \le M$ $\forall n$.

then for all k one has then $a_{n_k} \le M$ thus $(a_{n_k})_k$ is bounded. However: do the other claim.

bounded above

5) $\Rightarrow$ follows by 4) $\Leftarrow$ b_n bounded above $\exists M$ s.t $\forall n$ $b_n \le M$. Set $C := \max(a_1, \dots, a_N, M)$

then $\forall n$ one has that $a_n \le C$ (if $n \le N$ then trivially $a_n \le \max(a_1, \dots, a_N)$

if $n > N$ then $a_n \le M$)

6) If $(a_n)_n$ is increasing / strictly increasing / decreasing / strictly decreasing / monotone

6) PF: Let $(n_k)_k$ be a sequence of naturals and let $(b_n)_n := (a_{n_k})_k$

We have that $b_n = a_{n_k} \le a_{n_{k+1}} \le \dots \le a_{n_{k+n}} = b_{n+n}$, hence is increasing.

However other properties

II-2 ACCUMULATION POINTS

Definition. A real number x is an accumulation point of the sequence $(a_n)_{n\in\mathbb{N}}$ if every (or for every $\varepsilon>0$) the ε-neighbourhood of x contains infinitely many members of the sequence.

That is $a_n \in (x-\varepsilon, x+\varepsilon)$ for infinitely many n.

Alternatively, we can say $\forall \varepsilon>0 \ \forall N\in\mathbb{N} \ \exists n\geq N$ s.t. $|a_n - x| < \varepsilon$.

"FOR EACH $\varepsilon>0$, FOR EACH N NATURAL NUMBER THERE EXISTS $n\geq N$ S.T. $|a_n - x|<\varepsilon$"

or $a_n \in (x-\varepsilon, x+\varepsilon)$

Examples:

- The constant sequence (a_n) $a_n = C$ has exactly one accumulation point, namely $x=C$. Indeed, LET $\varepsilon>0$, AND $N\in\mathbb{N}$ ONE CAN CHOOSE $n = N+2 \geq N$ WHICH SATISFIES $a_n = a_{N+1} = C \in (C-\varepsilon, C+\varepsilon)$.

- The sequence $a_n = n$ doesn't have any accumulation points. ASSUME $x\in\mathbb{R}$ IS ACCUMULATION POINT. SEE $\varepsilon=1$. By the Archimedean property $\exists N$ s.t. $N > x+1$. Then $\forall n \geq N$ $a_n = n \geq N > x+1 \Rightarrow a_n \notin (x-1, x+1)$

- THE sequence $a_n = (-1)^n$ HAS TWO ACCUMULATION POINTS
  $x=-1$ IS AN ACCUMULATION POINT. INDEED, LET $\varepsilon>0$ AND $N\in\mathbb{N}$ ONE CAN CHOOSE $n = 2N+1$, WHICH SATISFIES $n\geq N$ $a_n = (-1)^{2N+1} = -1 \in (-1-\varepsilon, -1+\varepsilon)$
  $x=1$ IS AN ACCUMULATION POINT. INDEED, LET $\varepsilon>0$ AND $N\in\mathbb{N}$ ONE CAN CHOOSE $n = 2N$, WHICH SATISFIES $n = 2N \geq N$ AND $a_n = (-1)^{2N} = 1 \in (1-\varepsilon, 1+\varepsilon)$

- THE SEQUENCE (a_n) $a_n = \frac{1}{n}$ HAS ONE ACCUMULATION POINT, NAMELY $x=0$. LET $\varepsilon>0$ AND $N\in\mathbb{N}$, THERE EXISTS $n\in\mathbb{N}$ S.T.

$n > N$ AND $n > \frac{1}{\varepsilon}$, BY ARCHIMEDEAN PROPERTY

$(n > \max(N, \frac{1}{\varepsilon}))$

THEN $a_n = \frac{1}{n} < \varepsilon$. AND THUS $a_n \in (-\varepsilon, \varepsilon)$.

PROPOSITION: EVERY ACCUMULATION POINT OF A SUBSEQUENCE IS AN ACCUMULATION POINT OF THE SEQUENCE.

PROOF: LET $(a_n)_n$ BE A SEQUENCE AND $(b_m)_m$ BE A SUBSEQUENCE SO $b_k = a_{n_k}$ WHERE $n_1 < n_2 < \dots \in \mathbb{N}$ IS A STRICTLY INCREASING SEQUENCE. LET x BE AN ACC. POINT OF (b_k). LET $\varepsilon > 0$ AND LET $N \in \mathbb{N}$. THEN $\exists k \geq N$ s.t. $b_k \in (x-\varepsilon, x+\varepsilon)$. BUT $b_k = a_{n_k}$ AND $n_k \geq k \geq N$ SATISFIES $a_{n_k} \in (x-\varepsilon, x+\varepsilon)$

EXAMPLE: $(-2)^n$ HAS TWO SUBSEQUENCES $b_n = 1^{2n}$ & $c_n = (-2)^{2n+1}$

PROPOSITION: A SEQUENCE AND ITS TAIL HAVE THE SAME ACCUMULATION POINTS.

PROOF: LET $(a_n)_n$ BE A SEQUENCE, AND LET $M \in \mathbb{N}$ BE ITS TAIL $b_n = a_{M+n-1}$. SINCE (b_n) IS A SUBSEQUENCE OF (a_n) EVERY ACCUMULATION POINT OF (b_n) IS AN ACCUMULATION POINT OF (a_n). LET x BE AN ACCUMULATION POINT OF (a_n). LET $\varepsilon > 0$, $N \in \mathbb{N}$ THERE EXISTS $m \geq M+N-1$ SUCH THAT $a_m \in (x-\varepsilon, x+\varepsilon)$

SET $n = m - M + 1$. THEN $n \geq N$ AND

$b_n = a_{M+n-1} = a_m \in (x-\varepsilon, x+\varepsilon)$.

THUS EVERY ACCUMULATION POINT OF (a_n) IS AN ACCUMULATION POINT OF (b_n).

LEMMA: (NESTED INTERVAL PROPERTY)

LET $I_n = [u_n, v_n]$ BE NONEMPTY CLOSED INTERVALS, $(u_n, v_n \in \mathbb{R})$, SATISFYING $\forall n \in \mathbb{N}$ $I_{n+1} \subseteq I_n$

