THEN THERE EXISTS $a \in \mathbb{R}$ BELONGING TO EVERY I_n.

PROOF: THE FACT THAT $I_{n+1} \subseteq I_n$ IMPLIES

$u_n \leq u_{n+1} \leq v_{n+1} \leq v_n$

THE SET $S := \{u_n, n \in \mathbb{N}\}$ IS NON-EMPTY AND BOUNDED ABOVE BY v_1. DEFINE $a = \sup S$.

FIX $n \in \mathbb{N}$. SINCE $u_n \in S$, WE HAVE $u_n \leq a$.

WE NOW SHOW THAT v_n IS AN UPPER BOUND FOR S. LET $k \in \mathbb{N}$

IF $k \leq n$, THEN $u_k \leq u_n \leq v_n$.

IF $k > n$ THEN $I_k \subseteq I_n$ SO $u_k \leq v_n$

THUS $u_k \leq v_n$ FOR EVERY k. SINCE a IS THE LEAST UPPER BOUND OF S, WE OBTAIN $a \leq v_n$

THEREFORE $u_n \leq a \leq v_n$, SO $a \in I_n$. AS n WAS ARBITRARY CHOSEN, WE DEDUCE a BELONGS TO EVERY INTERVAL.

THEOREM: EVERY BOUNDED SEQUENCE OF REAL NUMBERS HAS AT LEAST ONE ACCUMULATION POINT.

PROOF: LET (a_n) BE A BOUNDED SEQUENCE.

$\exists\, m, M$ SUCH THAT $\forall n \in \mathbb{N}: m \leq a_n \leq M$. SET $u_1 := m$, $v_1 := M$

THUS $\forall N \in \mathbb{N}\ \exists n \geq N$ SUCH THAT $a_n \in I_1$

BISECT I_1 INTO TWO HALVES:

SET $c_1 = \frac{u_1 + v_1}{2}$, $I_1' := [u_1, c_1]$ AND $I_1'' := [c_1, v_1]$

AT LEAST ONE OF THE HALVES I_1' OR I_1'' HAS THE SAME PROPERTY AS I_1

$\forall N \in \mathbb{N}\ \exists M \geq N$ SUCH THAT $a_n \in I_1'$

OR $\forall N \in \mathbb{N}\ \exists n \geq N$ S.T. $a_n \in I_1''$.

INDEED, IF NEITHER HALF DID THERE WOULD BE INDICES

N_1 AND N_2 SUCH THAT $n \geq N_1 \Rightarrow a_n \notin I_1'$

$n \geq N_2 \Rightarrow a_n \notin I_1''$

THEN NO TERM WITH INDEX $n \geq \max(N_1, N_2)$ WOULD BELONG TO $I_1 = I_1' \cup I_1''$, A CONTRADICTION.

CHOOSE HALF WITH THE PROPERTY, CALL IT I_2.

LET u_2, v_2 BE SUCH THAT $I_2 = [u_2, v_2]$.

SET $c_2 := \frac{u_2 + v_2}{2}$. BISECT I_2 INTO TWO HALVES.

AT LEAST ONE OF THE HALVES I_2' AND I_2'' HAS THE SAME PROPERTY AS I_2:

$\forall N \in \mathbb{N} \ \exists n \geq N$ s.t. $a_n \in I_2'$ CHOOSE THE HALF CALL IT I_2 AND LET u_2, v_2 $I_2 = [u_2, v_2]$...

OR $\forall N \in \mathbb{N} \ \exists m \geq N$ s.t. $a_m \in I_2''$.

BY REPEATING THE PROCEDURE, WE OBTAIN NESTED INTERVALS

$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$. THE LENGTH OF I_k IS $\frac{M-m}{2^k}$.

BY NESTED INTERVAL PROPERTY $\exists a \in I_n \ \forall n \in \mathbb{N}$

LET $\varepsilon > 0$ AND $k \in \mathbb{N}$. CHOOSE k S.T. $\frac{M-m}{2^{k-1}} < \varepsilon$.

$\Leftrightarrow \frac{1}{2^{k-1}} < \frac{\varepsilon}{M-m} \Leftrightarrow \frac{M-m}{\varepsilon} < 2^{k-1}$ WHICH EXISTS BY THE ARCHIMEDEAN PROPERTY.

BY THE PROPERTY OF I_k THERE EXISTS $n \geq N$ WITH $a_n \in I_n$

SINCE $a \in I_k$ TOO, WE HAVE THAT

$$|a_n - a| < \text{length}(I_k) = \frac{M-m}{2^{k-1}} < \varepsilon.$$

HENCE $a_n \in (a-\varepsilon, a+\varepsilon)$. THUS a IS AN ACCUMULATION POINT FOR (a_n).

II-3 LIMITS

II-3-1 DEFINITION

DEF A SEQUENCE $(a_n)_n$ converges to $\ell \in \mathbb{R}$ if

ε-NGBD of ℓ $\Leftrightarrow$ $|a_n - \ell| < \varepsilon$

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad a_n \in (\ell - \varepsilon, \ell + \varepsilon)$$

"FOR ANY $\varepsilon > 0$, THERE EXISTS AN INDEX N S.T. FOR ALL INDICES n WHICH ARE LARGER THAN OR EQUAL TO N, THE MEMBER a_n BELONGS TO THE INTERVAL $(\ell - \varepsilon, \ell + \varepsilon)$."

ℓ IS A LIMIT IF FOR EVERY $\varepsilon > 0$, THERE EXISTS AN INDEX N SUCH THAT ALL MEMBERS OF THE SEQUENCE WITH INDEX AT LEAST N, BELONG TO $(\ell - \varepsilon, \ell + \varepsilon)$

WE WRITE $\ell = \lim_{n \to \infty} a_n$ AND SAY THAT ℓ IS THE LIMIT OF THE SEQUENCE a_n.

EXAMPLE: CONSIDER THE SEQUENCE $a_n = \frac{1}{n}$.

THE SEQUENCE IS $\underset{a_1}{1}, \underset{a_2}{\frac{1}{2}}, \underset{a_3}{\frac{1}{3}}, \ldots$

DISCUSSION WE SHOW THAT LIMIT IS 0.

TAKE $\varepsilon = 1$. EVERY TERM STARTING WITH THE TERM OF INDEX $N = 2$ SATISFIES $0 < \frac{1}{n} < 1$, HENCE BELONGS TO $(-1, 1)$

($\frac{1}{n} = a_n$)

TAKE $\varepsilon = \frac{1}{10} = 0{,}1$ EVERY TERM STARTING WITH THE TERM OF INDEX $N = 11$ SATISFIES $0 < a_n < \frac{1}{10}$, HENCE $a_n \in (-\frac{1}{10}, \frac{1}{10})$

($\frac{1}{n} = a_n$)

$\varepsilon = \frac{1}{100} = 0{,}01$ EVERY TERM STARTING ——— $N = 101$

SATISFIES $0 < a_n < \frac{1}{100}$, HENCE $a_n \in (-\frac{1}{100}, \frac{1}{100})$

($\varepsilon = \frac{1}{100}$ ...)

FORMAL PROOF LET $\varepsilon > 0$. BY ARCHIMEDEAN PROPERTY, THERE EXISTS $N \in \mathbb{N}$ SUCH THAT $\frac{1}{N} < \varepsilon$ (BECAUSE THIS IS EQUIVALENT TO $N > \frac{1}{\varepsilon}$)

FOR EVERY $n \geq N$, WE HAVE THAT $|a_n - 0| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$

THUS $a_n \in (-\varepsilon, \varepsilon)$

HENCE $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$. ■

FACT = A LIMIT OF A SEQUENCE IS ITS ACCUMULATION POINT.

THEOREM: THEOREM. IF A SEQUENCE CONVERGES, THEN IT HAS PRECISELY ONE ACCUMULATION POINT (NAMELY, ITS LIMIT)

PROOF: LET $(a_n)_{n\in\mathbb{N}}$ BE A CONVERGENT SEQUENCE.

LET $\lim_{n\to\infty} a_n = \ell$. THEN ℓ IS AN ACCUMULATION POINT FOR a_n, HENCE (a_n) HAS AT LEAST ONE ACCUMULATION POINT.

SUPPOSE IT HAS ANOTHER ACCUMULATION POINT, CALL IT b, $b \neq \ell$

BY THE DEFINITION OF CONVERGENCE, $\forall \varepsilon > 0$ THERE EXISTS $N \in \mathbb{N}$ SUCH THAT $a_n \in (\ell-\varepsilon, \ell+\varepsilon) \iff |a_n - \ell| < \varepsilon$.



CHOOSE $\varepsilon = |\ell - b|/3$. THERE EXISTS $N_0 \in \mathbb{N}$

S.T. $n \geq N_0$ ONE HAS THAT $a_n \in (\ell-\varepsilon, \ell+\varepsilon) \iff |a_n - \ell| < \varepsilon$

AS b IS AN ACCUMULATION POINT THERE EXISTS $m \geq N$ S.T.

$|a_m - b| < \varepsilon$.

THEN $|a_m - \ell| < \varepsilon$ AND $|a_m - b| = |b - a_m| < \varepsilon$.

IMPLY $|a_m - \ell + b - a_m| < 2\varepsilon \Rightarrow 3\varepsilon < 2\varepsilon \iff \varepsilon < 0$

$|a_m - \ell + b - a_m| = |b - \ell| = 3\varepsilon$.

BUT $\varepsilon > 0$, WE OBTAIN A CONTRADICTION. HENCE THERE IS NO SUCH ACCUMULATION POINT b.

COROLLARY: A CONVERGENT SEQUENCE HAS EXACTLY ONE LIMIT. IT DOES, IT HAS AT LEAST ONE.

PROOF: EVERY LIMIT OF A SEQUENCE IS AN ACCUMULATION POINT. SINCE A CONVERGENT SEQUENCE HAS EXACTLY ONE ACCUMULATION POINT, IT CANNOT HAVE TWO LIMITS.

$(a_n)_n$ HAS A LIMIT WE SAY $(a_n)_n$ IS CONVERGENT

$(a_n)_n$ DOES NOT HAVE A LIMIT, WE SAY $(a_n)_n$ IS DIVERGENT.

EXAMPLES: $a_n = (-1)^n$ IS DIVERGENT, BECAUSE IT HAS TWO ACCUMULATION POINTS

(MONOTONE CONVERGENCE THEOREM)

THEOREM: 1) AN INCREASING SEQUENCE IS CONVERGENT IFF IT IS BOUNDED ABOVE (IN THAT CASE), ITS LIMIT IS $\sup\{a_n \mid n \in \mathbb{N}\}$

("IF AND ONLY IF")

2) A DECREASING SEQUENCE IS CONVERGENT IFF IT IS BOUNDED BELOW (IN THAT CASE). ITS LIMIT IS $\inf\{a_n \mid n \in \mathbb{N}\}$

$\Rightarrow$

PROOF: 1) SUPPOSE FIRST (a_n) IS CONVERGENT AND LET ℓ BE ITS LIMIT. SET $\varepsilon = 1$, THERE EXISTS $N \in \mathbb{N}$ SUCH THAT $n \geq N \Rightarrow a_n \in (\ell - \varepsilon, \ell + \varepsilon) \Rightarrow a_n < \ell + \varepsilon$ ("IMPLIES")

AN UPPER BOUND FOR THE SEQUENCE IS $\ell + \varepsilon$ (FOR $n \geq N$ $a_n < \ell + \varepsilon$; FOR $n = 1, \dots, N-1$ $a_n \leq a_N < \ell + \varepsilon$)

$\Leftarrow$ SUPPOSE THAT (a_n) IS BOUNDED ABOVE.

LET $\ell = \sup\{a_n \mid n \in \mathbb{N}\}$ LET $\varepsilon > 0$. AS $\ell - \varepsilon$ IS NOT AN UPPER BOUND FOR $\{a_n \mid n \in \mathbb{N}\}$, THERE EXISTS $N \in \mathbb{N}$ S.T. $a_N > \ell - \varepsilon$. FOREVERY $n \geq N$, WE HAVE THAT $a_n > \ell - \varepsilon$ AND $a_n \leq \ell < \ell + \varepsilon$. IT FOLLOWS $a_n \in (\ell - \varepsilon, \ell + \varepsilon)$

HENCE (a_n) CONVERGES AND ITS LIMIT IS ℓ.

2) EXERCISE.

EXAMPLE: $a_n = n$ IS AN INCREASING SEQUENCE, NOT BOUNDED ABOVE. HENCE IT IS DIVERGENT.

(FACT: (FORMERLY NO) EVERY SUBSEQUENCE OF A CONVERGING SEQUENCE IS CONVERGENT AND IT CONVERGES TO THE SAME LIMIT.)

THEOREM: LET $(a_n)_n$ BE A SEQUENCE. THEN IT IS CONVERGENT IFF IT IS BOUNDED AND HAS PRECISELY ONE ACCUMULATION POINT.

$\Rightarrow$

PROOF: ASSUME $(a_n)_n$ IS CONVERGENT. LET $\ell := \lim_{n\to\infty} a_n$.

LET US CHOOSE $\varepsilon = 1$. THERE EXISTS $N \in \mathbb{N}$ S.T.

$n \geq N \Rightarrow a_n \in (\ell-1, \ell+1)$. WE SET $C := \max(a_1, \dots, a_{N-1}, \ell+1)$.

LET US CHOOSE $c := \min(a_1, \dots, a_{N-1}, \ell-1)$.

THEN $\forall n \in \mathbb{N}$ ONE HAS THAT $c \leq a_n \leq C$.

(FOR $n = 1, \dots, N-1$, WE HAVE $a_n \leq \max(a_1, \dots, a_{N-1}) \leq C$ AND $a_n \geq \min(a_1, \dots, a_{N-1}) \geq c$

FOR $n \geq N$, WE HAVE : $a_n < \ell+1 \leq C$

AND $a_n > \ell-1 \geq c$).

BY PREVIOUS THEOREM, $(a_n)_n$ HAS PRECISELY ONE ACCUMULATION POINT.

$\Leftarrow$ ASSUME $(a_n)_n$ IS BOUNDED AND HAS PRECISELY ONE ACCUMULATION POINT. CALL THIS POINT ℓ. SUPPOSE THAT ℓ IS NOT A LIMIT OF $(a_n)_n$. THIS MEANS PRECISELY THAT THERE EXISTS ε_0 S.T. $\forall N \in \mathbb{N}$ $\exists n \geq N$ WITH $a_n \notin (\ell-\varepsilon, \ell+\varepsilon)$. (*)

CHOOSE AN INDEX $n_1 \geq N$ SUCH THAT $a_{n_1} \notin (\ell-\varepsilon, \ell+\varepsilon)$.

NOW CONSIDER $N = n_1+1$. $\exists n \geq N$ WITH $a_n \notin (\ell-\varepsilon, \ell+\varepsilon)$

CHOOSE AN INDEX $n_2 \geq N$ (i.e. $n_2 > n_1$) SUCH THAT $a_{n_2} \notin (\ell-\varepsilon, \ell+\varepsilon)$

CONTINUE THE PROCESS HAVING CHOSEN n_k, APPLY (*) WITH $N = n_k+1$ TO DEFINE $n_{k+1} \geq N = n_k+1$. $b_k \notin (\ell-\varepsilon, \ell+\varepsilon)$

THIS DEFINES A SUBSEQUENCE $b_k = a_{n_k}$, FOR EVERY k. THE SEQUENCE $(b_k)_k$ IS BOUNDED, BECAUSE $(a_n)_n$ IS BOUNDED. IT HAS AN ACCUMULATION POINT b. EVERY ACCUMULATION POINT OF A SUBSEQUENCE, IS AN ACCUMULATION POINT OF THE ORIGINAL SEQUENCE. BY UNIQUENESS OF THE ACCUMULATION POINT FROM THE ASSUMPTION, $b = \ell$. HOWEVER, b CANNOT BE THE ACCUMULATION POINT OF b_k AS $b_k \notin (\ell-\varepsilon, \ell+\varepsilon)$

EXAMPLE: WE VERIFY THAT THE SEQUENCE $a_n = \frac{(-1)^n}{n}$ IS CONVERGENT. INDEED, IT IS BOUNDED. MORE $|a_n| \leq 1$ $\forall n \in \mathbb{N}$ BECAUSE $\frac{(-1)^n}{n} \leq 1$ IS TRUE AS $\frac{(-1)^n}{n} \leq \frac{1}{n} \leq 1$ FOR $n \in \mathbb{N}$

BOUNDED BELOW $a_n \geq -1$ $\forall n \in \mathbb{N}$, BECAUSE $\frac{(-1)^n}{n} \geq -1$ IS TRUE FOR n EVEN AS $\frac{(-1)^n}{n} \geq \frac{-1}{n} \geq -1$ IS TRUE FOR n ODD ($1 \leq n$)

IT IS SUFFICIENT TO SHOW THAT IT CANNOT HAVE TWO ACCUMULATION POINTS. LET US SHOW THAT IT CANNOT HAVE AN ACCUMULATION POINT DISTINCT FROM 0. ASSUME SUCH A POINT EXISTS, CALL IT ℓ. (and not a)

THERE EXISTS $N \in \mathbb{N}$, BY THE ARCHIMEDEAN PROPERTY SUCH THAT $|a_N| < \frac{|\ell|}{2}$ - INDEED $|a_N| = \frac{1}{N} < \frac{|\ell|}{2} \Leftrightarrow N > \frac{2}{|\ell|}$.

FOR $n \geq N$ $|a_n| \leq |a_N| < \frac{|\ell|}{2}$

THEN $|a_n - \ell| \geq ||a_n| - |\ell|| = ||\ell| - |a_n|| \geq \frac{|\ell|}{2}$.

THUS FOR $\varepsilon = \frac{|\ell|}{2}$ NO TERM WITH $n \geq N$ SATISFIES $|a_n - \ell| < \frac{|\ell|}{2} = \varepsilon$, IN OTHER WORDS $a_n \notin (\ell - \varepsilon, \ell + \varepsilon)$.

HENCE ℓ IS NOT AN ACCUMULATION POINT. THUS a_n HAS AT MOST ONE ACCUMULATION POINT. AS IT IS BOUNDED, IT HAS AT LEAST ONE ACCUMULATION POINT. BY PREVIOUS THEOREM, IT IS CONVERGENT.

EXERCISE: PROVE $\lim_{n \to \infty} a_n = 0$ BY DEFINITION.

EXAMPLE: THE SEQUENCE $1, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4, \frac{1}{5}, 5, \frac{1}{6}, 6, \ldots$

IS NOT CONVERGENT, DESPITE HAVING AN ACCUMULATION POINT. NAMELY, IT IS NOT BOUNDED.

ADD FACT FROM PREVIOUS LAST

II-3-2 OPERATIONS ON SEQUENCES AND

DEFINITION: LET $(a_n)_n$, $(b_n)_n$ BE REAL SEQUENCES AND LET $c \in \mathbb{R}$.
WE DEFINE THE FOLLOWING OPERATIONS:

- SUM $(a_n)_n + (b_n)_n := (a_n + b_n)_n$
- DIFFERENCE $(a_n)_n - (b_n)_n := (a_n - b_n)_n$
- SCALAR MULTIPLE $c \in \mathbb{R}$: $c \cdot (a_n)_n := (c a_n)_n$
- PRODUCT $(a_n)_n \cdot (b_n)_n := (a_n b_n)_n$
- QUOTIENT $(a_n)_n / (b_n)_n := (a_n / b_n)_n$ PROVIDED THAT $\forall n \in \mathbb{N}$ $b_n \neq 0$.

PROPOSITION: LET $(a_n)_n$ AND $(b_n)_n$ BE CONVERGENT SEQUENCES OF REAL NUMBERS, WITH $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$

THEN 1. SUM
$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = a + b$$

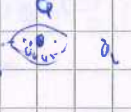
PRODUCT
2. $\lim_{n\to\infty} (a_n b_n) = ab$

3. FOR EVERY $c \in \mathbb{R}$: $\lim_{n\to\infty} (c a_n) = ca$.

4. DIFFERENCE $\lim_{n\to\infty} (a_n - b_n) = a - b$

5. QUOTIENT. ASSUMING $b \neq 0$ AND THAT $\forall n \in \mathbb{N}$, $b_n \neq 0$
THEN $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$.

PROOF: 1. LET $\varepsilon > 0$. THERE EXISTS $N_1, N_2 \in \mathbb{N}$ SUCH THAT



$$n \geq N_1 \Rightarrow |a_n - a| < \varepsilon/2$$
$$n \geq N_2 \Rightarrow |b_n - b| < \varepsilon/2$$

PUT $N := \max(N_1, N_2)$. FOR EVERY $n \geq N$ WE HAVE THAT

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \varepsilon.$$

(TRIANGLE INEQUALITY)

$\Rightarrow$ $(a_n + b_n)_n$ IS CONVERGENT AND $\lim_{n\to\infty} a_n + b_n = a + b$.

2. SINCE $(a_n)_n$ CONVERGES, IT IS BOUNDED. LET $M>0$ BE SUCH THAT $\forall n \in \mathbb{N} : |a_n| \leq M$.

WE WRITE $a_n b_n - ab = a_n(b_n - b) + b(a_n - a)$ AND HENCE

$$|a_n b_n - ab| \leq M|b_n - b| + |b| \cdot |a_n - a|$$

LET $\varepsilon > 0$. THERE EXISTS $N_1, N_2 \in \mathbb{N}$ SUCH THAT

$$n \geq N_1 \Rightarrow |a_n - a| < \frac{\varepsilon}{2(|b|+1)}$$

WE DIVIDE BY $|b|+1$ BECAUSE WE MULTIPLY BY b AND $|b| \not< |b|$

$$n \geq N_2 \Rightarrow |b_n - b| < \frac{\varepsilon}{2M}$$

THEN FOR $n \geq \max\{N_1, N_2\}$ WE HAVE THAT

$$|a_n b_n - ab| < M \cdot \frac{\varepsilon}{2M} + \frac{|b|}{|b|+1} \cdot \frac{\varepsilon}{2}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

3, 4, 7 EXERCISES

LEMMA

5. ASSUME $(b_n)_n$ IS A SEQUENCE SUCH THAT $b \neq 0$ AND $b_n \neq 0$ $\forall n$

THEN $\lim_{n\to\infty} \left|\frac{1}{b_n}\right| = \frac{1}{b}$.

$\exists N_0 \in \mathbb{N}$ S.T. $n \geq N_0 \Rightarrow |b_n - b| < \frac{|b|}{2}$. FOR EVERY $n \geq N_0$, BY TRIANGLE INEQUALITY $|b_n| \geq |b| - |b_n - b| > \frac{|b|}{2}$

CONSEQUENTLY: $\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b_n - b|}{|b_n| \cdot |b|} \leq \frac{2}{|b|^2} |b_n - b|$.

LET $\varepsilon > 0$ THERE EXISTS $N_1 \in \mathbb{N}$ SUCH THAT $n \geq N_1 \Rightarrow |b_n - b| < \frac{\varepsilon \cdot |b|^2}{2}$. FOR EVERY $n \geq \max(N_0, N_1)$, WE HAVE THAT

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| < \varepsilon$$

TO PROVE 5) WE COMBINE 2. AND LEMMA.

II-3-5 DIVERGENCE TO INFINITY

DEFINITION. LET (a_n) BE A SEQUENCE OF REAL NUMBERS (OR PLUS INFINITY).

WE SAY THAT (a_n) DIVERGES TO INFINITY AND WRITE

$$\lim_{n\to\infty} a_n = +\infty \quad \text{IF} \quad \forall M \in \mathbb{R}\ \exists N \in \mathbb{N}\ \forall n \geq N \quad a_n > M.$$

WE SAY THAT (a_n) DIVERGES TO MINUS INFINITY AND WRITE

$$\lim_{n\to\infty} a_n = -\infty \quad \text{IF} \quad \forall m \in \mathbb{R}\ \exists N \in \mathbb{N}.\ \forall n > N \quad a_n < m.$$

EXAMPLE: WE HAVE THAT $\lim_{n\to\infty} n^2 = +\infty$ (ARCHIMEDEAN PROPERTY)

· WE HAVE THAT $\lim_{n\to\infty} -n = -\infty$.

REMARK: IF $\lim_{n\to\infty} a_n = +\infty$ THEN a_n IS UNBOUNDED ABOVE.

IF (a_n) IS UNBOUNDED ABOVE, THEN IT MAY NOT DIVERGE TO $+\infty$

EXAMPLE $a_n = (1+(-1)^n)\cdot n$. IS UNBOUNDED ABOVE AS GIVEN $M>0$ $\exists n_0 > M$ BY ARC. PROPERTY THEN $a_{2n_0} = 2\cdot 2n_0 = 4n_0 > M$ (FOR $n = 2n_0$)

ON THE OTHER HAND FOR $M = 5$ $\forall N \in \mathbb{N}$ $\exists n \geq N$ S.T $a_n \leq M$

E.G. TAKE $n = 2N+1$. THEN $a_n = (1+(-1)^{2N+1})\cdot(2N+1) = 0\cdot(2N+1) = 0$

OPERATIONS WITH INFINITE LIMITS

THEOREM LET $(a_n)_n$ AND $(b_n)_n$ BE SEQUENCES OF REAL NUMBERS.

1) **SUMS** · IF $\lim_{n\to\infty} a_n = +\infty$ AND $\lim_{n\to\infty} b_n = +\infty$ (OR $\lim_{n\to\infty} b_n \geq +\infty$) THEN $\lim_{n\to\infty}(a_n+b_n) = +\infty$

— " — AND $\lim_{n\to\infty} b_n = b \in \mathbb{R}$ THEN $\lim_{n\to\infty}(a_n+b_n) = +\infty$

IF $\lim_{n\to\infty} a_n = -\infty$ AND $\lim_{n\to\infty} b_n = -\infty$, THEN $\lim_{n\to\infty}(a_n+b_n) = -\infty$

IF $\lim_{n\to\infty} a_n = -\infty$ AND $\lim_{n\to\infty} b_n = b \in \mathbb{R}$, THEN $\lim_{n\to\infty}(a_n+b_n) = -\infty$

2) **PRODUCTS** IF $\lim_{n\to\infty} a_n = +\infty$ AND $\lim_{n\to\infty} b_n = +\infty$ THEN $\lim_{n\to\infty}(a_n b_n) = +\infty$

— " — AND $\lim_{n\to\infty} b_n = b > 0$ THEN $\lim_{n\to\infty}(a_n b_n) = +\infty$

— " — AND $\lim_{n\to\infty} b_n = b < 0$ THEN $\lim_{n\to\infty}(a_n b_n) = -\infty$

IF $\lim_{n\to\infty} a_n = +\infty$ AND $\lim_{n\to\infty} b_n = -\infty$ THEN $\lim_{n\to\infty} a_n b_n = -\infty$

$\lim_{n\to\infty} a_n = -\infty$ AND —||— THEN $\lim_{n\to\infty} a_n b_n = +\infty$

IF —||— AND $\lim_{n\to\infty} b_n = b > 0$ THEN $\lim_{n\to\infty} a_n b_n = -\infty$

—||— AND $\lim_{n\to\infty} b_n = b < 0$ THEN $\lim_{n\to\infty} a_n b_n = +\infty$

RECIPROCALS

IF $\lim_{n\to\infty} b_n = b > 0$ [illegible] AND $b_n \neq 0$ $\forall n$ THEN $\lim_{n\to\infty} b_n = +\infty$

$b < 0$ $= -\infty$

IF $\lim_{n\to\infty} b_n = +\infty$ AND $\forall n \cdot b_n \neq 0$ THEN $\lim_{n\to\infty} \frac{1}{b_n} = 0$

—||— $= -\infty$ — ||— $= 0$

FORMULAS FOR QUOTIENTS IF $\lim_{n\to\infty} a_n = a \in \mathbb{R}$ AND $\lim_{n\to\infty} b_n = +\infty$ OR $\lim_{n\to\infty} b_n = -\infty$

THEN $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. A [illegible]

IF FURTHERMORE $a_n \neq 0$ $\forall n$ AND $a > 0$ THEN $\lim_{n\to\infty} \frac{b_n}{a_n} = +\infty$ IF $\lim_{n\to\infty} b_n = +\infty$

$\lim_{n\to\infty} \frac{b_n}{a_n} = -\infty$ IF $\lim_{n\to\infty} b_n = -\infty$

IF $a < 0$ THEN $\lim_{n\to\infty} \frac{b_n}{a_n} = -\infty$ IF $\lim_{n\to\infty} b_n = +\infty$ AND $\lim_{n\to\infty} \frac{b_n}{a_n} = +\infty$

IF $\lim_{n\to\infty} b_n = -\infty$.

REMARK: FORMS $+\infty + (-\infty)$, $0 \cdot (\pm\infty)$ AND $\frac{\pm\infty}{\pm\infty}$

GIVE NO GENERAL CONCLUSION. THEY ARE CALLED INDETERMINATE FORMULAS

EXAMPLE: $a_n = n$ $b_n = (-n)$ $a_n + b_n \to 0$ $a_n = n^2$, $b_n = -n$ $a_n + b_n \to +\infty$

PROOF: IF $\lim$ SUM OF TWO SEQUENCE:

$\lim_{n\to\infty} (a_n) = +\infty$, $\lim_{n\to\infty} (b_n) = b \in \mathbb{R}$

LET $M > 0$

$\exists N_1 \in \mathbb{N}$: $\forall n \geq N_1$: $a_n > M - b + 1$

FOR $\varepsilon = 1$ $\exists N_2 \in \mathbb{N}$: $\forall n \geq N_2$: $|b_n - b| < 1$, HENCE $b_n > b - 1$

THEN FOR $N = \max(N_1, N_2)$, WE HAVE THAT $a_n + b_n > M - b + 1 + b - 1 = M$