Analysis I
UP - FAMNIT
Fall 2026 November 04-05, 2026

Tutorial 5

1. Obtain $\liminf$ and $\limsup$ of the sequences below. Also, list their accumulation points and find a monotonic subsequence.

   (a) $a_n = \frac{1}{n}$ (c) $c_n = (-1)^n + \frac{3}{4n+1}$ (e) $e_n = n^{(-1)^n}$

   (b) $b_n = n^2$ (d) $d_n = 1 + (-1)^n$ f) $f_n = \sin\frac{n\pi}{2} + \cos\frac{n\pi}{2}$

2. Define a sequence (a_n) by

$$a_{2n-1} = n, \qquad a_{2n} = \frac{1}{n}, \qquad n \geq 1.$$

   (a) Write the first eight terms.

   (b) Is the sequence bounded?

   (c) Does it have any convergent subsequences?

   (d) Determine all its accumulation points (subsequential limits).

3. Consider

$$a_n = (-1)^n \left(1 + \frac{1}{n}\right), \qquad n \geq 1.$$

   (a) Write the first six terms.

   (b) Show that the sequence is bounded.

   (c) Determine its maximum and minimum.

   (d) Find all its accumulation points.

   (e) Does the sequence converge?

4. Consider

$$a_n = \frac{3n+1}{n+2}, \qquad n \geq 1.$$

   (a) Find the first four terms.

   (b) Prove that the sequence is strictly increasing.

   (c) Show that it is bounded and find its supremum and infimum.

   (d) Determine whether it converges.

Some Solutions

1. **Definition** Let (a_n) be a bounded sequence of real numbers.
   The **limit superior**, denoted by $\limsup_{n\to\infty} a_n$, is the largest accumulation point of the sequence.
   The **limit inferior**, denoted by $\liminf_{n\to\infty} a_n$, is the smallest accumulation point of the sequence.
   Remark. A bounded sequence (a_n) converges if and only if
   $$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

Sequence	Monotonic subsequences	Set of acc. points	lim inf	lim sup
$a_n = \frac{1}{n}$	(a_n)	$\{0\}$	0	0
$b_n = n^2$	(b_n)	$\{+\infty\}$	$+\infty$	$+\infty$
$c_n = (-1)^n + \frac{3}{4n+1}$	$s_{2k} = 1 + \frac{3}{4n+1}$, $s_{2k-1} = -1 + \frac{3}{4n+1}$, $k > 0$	$\{-1, 1\}$	-1	1
$d_n = 1 + (-1)^n$	$t_{2k} = 2$, $t_{2k-1} = 0$, $k > 0$	$\{0, 2\}$	0	2
$e_n = n^{(-1)^n}$	$y_{2k} = 2k$, $y_{2k-1} = \frac{1}{2k-1}$, $k > 0$	$\{0, +\infty\}$	0	$+\infty$
$f_n = \sin\frac{n\pi}{2} + \cos\frac{n\pi}{2}$	$(*)$	$\{-1, 1\}$	-1	1

   $(*)$ Observe that $\sin\frac{n\pi}{2}$ and $\cos\frac{n\pi}{2}$ are periodic. A full lap is completed anytime n gives four steps. Specifically:
   $$\begin{aligned}
   &n = 4k, && \sin 2k\pi + \cos 2k\pi = 0 + 1 && = 1\\
   &n = 4k+1, && \sin(2k\pi + \frac{\pi}{2}) + \cos(2k\pi + \frac{\pi}{2}) = 1 + 0 && = 1\\
   &n = 4k+2, && \sin(2k\pi + \pi) + \cos(2k\pi + \pi) = 0 - 1 && = -1\\
   &n = 4k+3, && \sin(2k\pi + \frac{3\pi}{2}) + \cos(2k\pi + \frac{3\pi}{2}) = -1 + 0 && = -1
   \end{aligned}$$

4. (a) The first four terms are
   $$a_1 = \frac{4}{3}, \qquad a_2 = \frac{7}{4}, \qquad a_3 = 2, \qquad a_4 = \frac{13}{6}.$$
   (b) Rewrite the terms as
   $$a_n = 3 - \frac{5}{n+2}.$$
   Therefore,
   $$a_{n+1} - a_n = \frac{5}{n+2} - \frac{5}{n+3} = \frac{5}{(n+2)(n+3)} > 0.$$
   Thus (a_n) is strictly increasing.

(c) Since (a_n) is increasing, $a_1 = \frac{4}{3}$ is its minimum. Moreover $a_n < 3$ for all n, and $a_n \to 3$ (as shown in part (d)), so 3 is its least upper bound. Thus

$$\inf\{a_n : n \in \mathbb{N}\} = \frac{4}{3}, \qquad \sup\{a_n : n \in \mathbb{N}\} = 3.$$

(d) Since $1/(n+2) \to 0$,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(3 - \frac{5}{n+2}\right) = 3.$$