MATHEMATICAL METHODS

LECTURE 6

1. Number sets

1.1. **Complex numbers.** Formally, one can define a *complex number* as an ordered pair $(a, b) \in \mathbb{R}^2$, where the binary operations $+$ and $\cdot$ are defined as $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$. It is not difficult to verify, that with these definitions for addition and multiplication, the set of all complex numbers $\mathbb{C}$ becomes a field, i.e., axioms (A), (M) and (D) are satisfied, where $(0, 0)$ and $(1, 0)$ are in the role of 0 and 1, respectively. If we define $\mathrm{i} = (0, 1)$, then $\mathrm{i}^2 = (0, 1) \cdot (0, 1) = (-1, 0) = -(1, 0) = -1$. Then $(a, b) = a + \mathrm{i}b$ for real a and b: $a + \mathrm{i}b = (a, 0) + (0, 1) \cdot (b, 0) = (a, 0) + (0, b) = (a, b)$.

The preceeding paragraph confirms the usual notation for a complex number: $z = x + \mathrm{i}y$, where x and y are real numbers, that are called the *real part* and the *imaginary part* of z; in symbols, $x = \Re\{z\}$ and $y = \Im\{z\}$. The complex number $\bar{z} = x - \mathrm{i}y$ is called the *conjugate* of z, and $|z| = \sqrt{z \cdot \bar{z}} = \sqrt{x^2 + y^2} \geq 0$ is known as the *absolute value* of z. Note that if $x \in \mathbb{R}$, then $\bar{x} = x$ and

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

Some properties:

(1) $\overline{z + w} = \overline{z} + \overline{w}$;
(2) $\overline{zw} = \overline{z} \cdot \overline{w}$;
(3) $z + \overline{z} = 2\Re\{z\}$ and $z - \overline{z} = 2\mathrm{i}\Im\{z\}$;
(4) $|z| = 0$ if and only if $z = 0$;
(5) $|\bar{z}| = |z|$;
(6) $|zw| = |z| \cdot |w|$;
(7) $|\Re\{z\}| \leq |z|$ and $|\Im\{z\}| \leq |z|$;
(8) $|z + w| \leq |z| + |w|$.

Every complex number can be written in the following *polar* form:

$$z = |z| \left(\cos\varphi + \mathrm{i}\sin\varphi\right),$$

where $\varphi \in \mathbb{R}$ is the *argument* of z. Of course,

$$\cos\varphi + \mathrm{i}\sin\varphi = \cos\left(\varphi + 2k\pi\right) + \mathrm{i}\sin\left(\varphi + 2k\pi\right)$$

for every $k \in \mathbb{Z}$, so the argument of z is not uniquely determined, but it is unique "modulo" 2π. If $z = |z| \left(\cos\varphi_1 + \mathrm{i}\sin\varphi_1\right)$ and $w = |w| \left(\cos\varphi_2 + \mathrm{i}\sin\varphi_2\right)$, then

$$z \cdot w = |z||w| \left(\cos\left(\varphi_1 + \varphi_2\right) + \mathrm{i}\sin\left(\varphi_1 + \varphi_2\right)\right).$$

In particular, if $z = |z| \left(\cos\varphi + \mathrm{i}\sin\varphi\right)$, then $z^n = |z|^n \left(\cos\left(n\varphi\right) + \mathrm{i}\sin\left(n\varphi\right)\right)$ for $n \in \mathbb{N}$.

2. Numerical sequences

A (real) sequence is a map from $\mathbb{N}$ to $\mathbb{R}$, $f\colon \mathbb{N} \to \mathbb{R}$, where we usually denote this map as $f(n) = a_n$. The values of this map, a_n, are known as terms of a sequence $a_1, a_2, a_3, \ldots$, where the short notation for this sequence is $\{a_n\}_{n=1}^{\infty}$ or $(a_n)_{n=1}^{\infty}$. We say that $\{a_n\}_{n=1}^{\infty}$ is bounded above (resp. below), if the corresponding *set* $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ is bounded above (resp. below), see Lecture 4. Then a sequence is bounded, if it is bounded above *and* below. Similarly, we define the least upper bound of the sequence as $\sup(a_n) = \sup\{a_1, a_2, \ldots\}$, and the greatest lower bound of the sequence as $\inf(a_n) = \inf\{a_1, a_2, \ldots\}$. Finally, concerning the growth of terms, we also define the following:

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(1) $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ is *increasing*, if $a_{n+1} \geq a_n$ for every $n \in \mathbb{N}$. In addition, the sequence is *strictly increasing*, if $a_{n+1} > a_n$ for every $n \in \mathbb{N}$.
(2) $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ is *decreasing*, if $a_{n+1} \leq a_n$ for every $n \in \mathbb{N}$. In addition, the sequence is *strictly decreasing*, if $a_{n+1} < a_n$ for every $n \in \mathbb{N}$.
(3) $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ is *monotonic*, if the sequence is increasing or decreasing.

Example 1. Let $a_n = 3n/(4n+1)$. The first five terms are: $a_1 = 3/5 = 0.6$, $a_2 = 2/3 = 0.\overline{6}$, $a_3 = 9/13 \approx 0.692$, $a_4 = 12/17 \approx 0.706$ and $a_5 = 5/7 \approx 0.714$. We have a feeling that this sequence is strictly increasing. Indeed, the proof would be:

$$a_{n+1} - a_n = \frac{3(n+1)}{4(n+1)+1} - \frac{3n}{4n+1} = \frac{3(n+1)(4n+1) - 3n(4n+5)}{(4n+5)(4n+1)} = \frac{3}{(4n+5)(4n+1)} > 0$$

for every $n \in \mathbb{N}$. With this we immediately have $\inf(a_n) = a_1 = 3/5$, so this sequence is bounded below. It is also bounded above, as we can easily see from

$$a_n = \frac{3n}{4n+1} = \frac{3}{4+\frac{1}{n}} < \frac{3}{4}$$

for every $n \in \mathbb{N}$ since then $4 + 1/n > 4$. Using the concept of the *limit*, we can actually show that $3/4$ is also the least upper bound of the sequence, that is, $\sup(a_n) = 3/4$.

We say that a sequence $\{a_n\}_{n=1}^{\infty}$ *converges to* $a \in \mathbb{R}$, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|a_n - a| \leq \varepsilon$ for every $n \geq n_0$. If a sequence converges, then we say that the sequence is *convergent*. The *limit* of $\{a_n\}_{n=1}^{\infty}$ is such a number a for which the sequence converges to: $\lim_{n\to\infty} a_n = a$, also $a_n \to a$. *Divergent sequence* is such a sequence that does not converge.

Example 2. Consider again $a_n = 3n/(4n+1)$. Then

$$\left| a_n - \frac{3}{4} \right| = \frac{3}{4(4n+1)}.$$

Take an arbitrary $\varepsilon > 0$. Then choose $n_0 \in \mathbb{N}$ such that $n_0 \geq 1/\varepsilon$. For $n \geq n_0$ we then have $4(4n+1) \geq 3n_0 \geq 3/\varepsilon$, which gives $3/(4(4n+1)) \leq \varepsilon$. Therefore, $\lim_{n\to\infty} a_n = 3/4$ by definition of the limit. Because the sequence is increasing, we then have $\sup(a_n) = \lim_{n\to\infty} a_n = 3/4$.

Example 3. Consider the sequence which is given as $a_n = (-1)^n$. This sequence is clearly not monotonic. Although $\inf(a_n) = -1$ and $\sup(a_n) = 1$, its limit does not exist.

Here we list some common rules for the evaluation of limits:

(1) If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are convergent sequences, then:
   (a) $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$;
   (b) $\lim_{n\to\infty} (\lambda \cdot b_n) = \lambda \cdot \lim_{n\to\infty} a_n$ for a constant $\lambda \in \mathbb{R}$;
   (c) $\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$;
   (d) If $b_n \neq 0$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} b_n \neq 0$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n};$$

(2) $\lim_{n\to\infty} 1/n = 0$;
(3) $\lim_{n\to\infty} a^n = 0$, if $|a| < 1$;
(4) $\lim_{n\to\infty} n^k/a^n = 0$, if $k \in \mathbb{R}$ and $a > 1$.