MATHEMATICAL METHODS

LECTURE 7

1. Numerical sequences

We say that $\{a_n\}_{n=1}^{\infty}$ *diverges to* ∞, if for every $A \in \mathbb{R}$ there exists $n_0 \in \mathbb{N}$ such that $a_n \geq A$ for every $n \geq n_0$. In this case we write $\lim_{n\to\infty} a_n = \infty$, also $a_n \to \infty$. On the other hand, we say that $\{a_n\}_{n=1}^{\infty}$ *diverges to* $-\infty$, if for every $A \in \mathbb{R}$ there exists $n_0 \in \mathbb{N}$ such that $a_n \leq A$ for every $n \geq n_0$. In this case we write $\lim_{n\to\infty} a_n = -\infty$, also $a_n \to -\infty$.

Example 1. Consider $a_n = (n^2 - 2)/(n + 1)$. First, we have

$$a_n = \frac{n - \frac{2}{n}}{1 + \frac{1}{n}} \geq \frac{n - 1}{1 + \frac{1}{n}} \geq \frac{2}{3}(n - 1)$$

for every $n \geq 2$ since $2/n \leq 1$, $1 + 1/n \leq 3/2$ and $n - 1 \geq 0$ for every $n \geq 2$. Take an arbitrary $A \in \mathbb{R}$. Then choose $n_0 \in \mathbb{N}$ such that $n_0 \geq \max\{(3/2)A + 1, 2\}$. Then $(2/3)(n - 1) \geq A$ for all $n \geq n_0$, which implies that $a_n \geq A$ for all $n \geq n_0$. Therefore, $a_n \to \infty$. This is a formal proof. However, one can quickly see this from the above, since $\lim_{n\to\infty}(n - 2/n) = \lim_{n\to\infty} n = \infty$, while $\lim_{n\to\infty}(1 + 1/n) = 1$.

2. Infinite series

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. We can sum its terms: $a_1 + a_2 + \cdots + a_k = \sum_{j=1}^{k} a_j$. We are making a new sequence of *partial sums* during such a process: $\{s_k\}_{k=1}^{\infty}$, where $s_k = \sum_{j=1}^{k} a_j$. If this sequence converges to $s \in \mathbb{R}$, i.e., $s_k \to s$, then we call its limit the *sum of infinite series* $\sum_{n=1}^{\infty} a_n$ and write

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} s_k = s.$$

The series $\sum_{n=1}^{\infty} a_n$ is then *convergent*. In the opposite case, i.e., when $\{s_k\}_{k=1}^{\infty}$ diverges, we call the series *divergent*.

2.1. **Geometric series.** Let $a \in \mathbb{R}$ and $|q| < 1$. Then the series $\sum_{n=1}^{\infty} aq^{n-1}$ is called the *geometric series*. It is easy to see that its partial sums equal

$$s_k = a\left(1 + q + q^2 + \cdots + q^{k-1}\right) = a\frac{1 - q^k}{1 - q}. \tag{1}$$

Therefore, $s_k \to a/(1 - q)$, implying

$$\sum_{n=1}^{\infty} aq^{n-1} = \frac{a}{1 - q} \quad \text{for} \quad a \in \mathbb{R} \quad \text{and} \quad |q| < 1.$$

Equation (1) for an evaluation of a *geometric sum*, which is true for *any* $q \in \mathbb{R}$, is particularly useful in relation to interest calculations.

Example 2. At the beginning of 2026 we had 80000 EUR on our banking account. Then we start making deposits of 4000 EUR at the beginning of each year that follows. Our plan is to make a final deposit at the beginning of 2036. How much money we will have in the account just after this final deposit? Yearly interest rate is 4%.

Solution: Denote by G_0 the initial amount of money, that is, $G_0 = 80000$ EUR, and by a the amount of yearly deposits, that is, $a = 4000$ EUR. In addition, let $q = 1 + p/100$, where $p/100$ is with $p = 4$ bank's yearly interest rate, so $q = 1.04$ in our case. Then $qG_0 + a$ is the amount of

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money on our banking account at the beginning of 2027 and just after our first deposit. Similarly, $q(qG_0+a)+a = q^2G_0+(q+1)a$ is the amount of money on our banking account at the beginning of 2028 and just after our second deposit. Observing the pattern, we have (also using (1))

$$q^nG_0 + \left(q^{n-1}+q^{n-2}+\cdots+q+1\right)a = q^nG_0 + a\frac{q^n-1}{q-1}$$

amount of money on our banking account at the beginning of $2026+n$ and just after our n-th deposit. Using numerical values for G_0, q, a and $n=10$, we calculate by the above formula that we will have around 166444 EUR in our bank account just after the final deposit at the beginning of 2036.

3. Differentiation

First, some discussion about the concept of a *limit of a function*. Let $(a,b)\subset\mathbb{R}$ be an open interval and $x_0\in(a,b)$. Assume that $f\colon (a,b)\setminus\{x_0\}\to\mathbb{R}$ is given. Then we say that $y\in\mathbb{R}$ is a *limit of f at x_0*, if for every $\varepsilon>0$ there exists $\delta>0$, such that $|f(x)-y|\leq\varepsilon$ for all $x\in(a,b)$ with $0<|x-x_0|\leq\delta$. This can be written with quantifiers as

$$\forall\varepsilon>0.\ \exists\delta>0.\ \forall x.\ \left(0<|x-x_0|\leq\delta \implies |f(x)-y|\leq\varepsilon\right).$$

If such y exists, then we denote it as $\lim_{x\to x_0} f(x)$. Note the striking similarity with the definition of a continuity; there, we allow also $x=x_0$, while the limit (if it exists) at x_0 does not in general depend on a (possible since f does not need to be defined at x_0) value of f at x_0. Saying so, if f is continuous on (a,b), then $\lim_{x\to x_0} f(x) = f(x_0)$.

Example 3. Define $f\colon\mathbb{R}\to\mathbb{R}$ as

$$f(x) = \begin{cases} 1, & x\in\mathbb{R}\setminus\{0\},\\ 2, & x=0.\end{cases}$$

This function is not continuous at $x=0$. However, $\lim_{x\to0} f(x)=1$ and $f(0)=2$. In fact, for an arbitrary $c\in\mathbb{R}$ we could define $f(0)=c$, and still $\lim_{x\to0} f(x)=1$, with the same outcome even if we let f being undefined at $x=0$. Less trivial example of the latter would be $g(x)=\frac{\sin x}{x}$, which is continuous on $\mathbb{R}\setminus\{0\}$ but undefined at $x=0$. Later, we are going to show that $\lim_{x\to0} g(x)=1$.

Let $f\colon(a,b)\to\mathbb{R}$, where $(a,b)\subset\mathbb{R}$. We say that f is *differentiable at $x\in(a,b)$*, if the following limit

$$\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

exists. If it exists, then we denote its value by $f'(x)$, and call it the *derivative of f at x*. We say that f is *differentiable on* (a,b), if it is differentiable at every $x\in(a,b)$. Geometric interpretation of the derivative is that $f'(x)$ is the slope coefficient of the tangent on the graph of f at the point $(x,f(x))$. This is because $(f(x+h)-f(x))/h$ is the slope coefficient of the secant through two points $(x,f(x))$ and $(x+h,f(x+h))$, and this line should in principle coincide with the tangent at $(x,f(x))$ when $h\to0$. While f is continuous at x if f is differentiable at x, the converse is not true.

Example 4. Consider $f(x)=|x|$. Then f is not differentiable at $x=0$:

$$\frac{f(0+h)-f(0)}{h} = \frac{|h|}{h} = \begin{cases} 1, & h>0,\\ -1, & h<0\end{cases}$$

so the limit of the above function at 0 does not exist.