IF $\lim_{n\to\infty} a_n = +\infty$ AND $\lim_{n\to\infty} b_n = -\infty$ THEN $\lim_{n\to\infty} a_n b_n = -\infty$

$\lim_{n\to\infty} a_n = -\infty$ AND —||— THEN $\lim_{n\to\infty} a_n b_n = +\infty$

~~SIM~~ IF —||— AND $\lim_{n\to\infty} b_n = b > 0$ THEN $\lim_{n\to\infty} a_n b_n = -\infty$

—||— AND $\lim_{n\to\infty} b_n = b < 0$ THEN $\lim_{n\to\infty} a_n b_n = +\infty$

RECIPROCALS ~~QUOTIENTS~~

~~IF $\lim_{n\to\infty} b_n = 0$ AND $b_n > 0$ $\forall n$ THEN $\lim_{n\to\infty} \frac{1}{b_n} = +\infty$~~

IF $\lim_{n\to\infty} b_n = +\infty$ AND $\forall n \cdot b_n \neq 0$ THEN $\lim_{n\to\infty} \frac{1}{b_n} = 0$

—||— $= -\infty$ — $\frac{1}{\quad}$ —— $= 0$

FORMULAS FOR QUOTIENTS IF $\lim_{n\to\infty} a_n = a \in \mathbb{R}$ AND $\lim_{n\to\infty} b_n = +\infty$ OR

THEN $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

page 1 , IF FURTHERMORE $a_n \neq 0$ AND $a > 0$ $\forall n$ THEN $\lim_{n\to\infty} \frac{b_n}{a_n} = +\infty$ IF $\lim$

$\lim_{n\to\infty} \frac{b_n}{a_n} = -\infty$ IF $\lim$

IF $a < 0$ THEN $\lim_{n\to\infty} \frac{b_n}{a_n} = -\infty$ IF $\lim b_n$ AND $\lim$

IF $\lim$

REMARK: FORMS $+\infty + (-\infty)$, $0 \cdot (\pm\infty)$ AND $\frac{\pm\infty}{\pm\infty}$

DON'T GIVE NO GENERAL CONCLUSION. THEY ARE CALLED INDETERMINATE

EXAMPLE: $a_n = n$ $b_n = (-n)$ $a_n + b_n \to 0$ $a_n = n^2$, $b_n = -n$ $a_n + b_n \to$

PROOF: ~~IF (a_n)~~ SUM OF TWO SEQUENCES:

$\lim_{n\to\infty} (a_n) = +\infty$, $\lim_{n\to\infty} (b_n) = b \in \mathbb{R}$

LET $M > 0$

~~$\forall N_1 \in \mathbb{N}$~~ $\exists N_1 \in \mathbb{N}: \forall n \geq N_1: a_n > M - b + 1$

FOR $\varepsilon = 1$ $\exists N_2 \in \mathbb{N}: \forall n \geq N_2: |b_n - b| < 1$, HENCE $b_n > b - 1$

THEN FOR $N = \max\{N_1, N_2\}$, WE HAVE THAT $a_n + b_n > M - b + 1 + b - 1 = M$

MULTIPLICATION: LET $(a_n)_n$ BE $\lim_{n\to\infty} a_n = +\infty$ $\exists N_1$: $n \geq N_1$ $a_n > \max(M, 0)$. IF $\lim_{n\to\infty} b_n = +\infty$ $\exists N_2$ $n \geq N_2$ $b_n > 1$

$N = \max\{N_1, N_2\}$, $a_n b_n > a_n > M$ FOR $n \geq N$.

EXAMPLE: $\lim_{n\to\infty} \frac{n^2+2n+3}{n+1}$ "BAD APPROACH" $\lim_{n\to\infty} n^2+2n+3 = +\infty$

$\lim_{n\to\infty} n+1 = +\infty$ $\frac{+\infty}{+\infty}$ INDETERMINATE FORM.

CORRECT APPROACH

$n^2+2n+3 = (n+1)^2 + 2$ WE OBTAIN $\frac{n^2+2n+3}{n+1} = \frac{(n+1)^2+2}{n+1} = n+1+\frac{2}{n+1}$

NOW $\lim_{n\to\infty} (n+1) = \lim_{n\to\infty} n + \lim_{n\to\infty} 1 = +\infty + 1 = +\infty$.

$$\lim_{n\to\infty} \frac{2}{n+1} = \frac{\lim_{n\to\infty} 2}{\lim_{n\to\infty} n+1} = \frac{2}{+\infty} = 0$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{n^2+2n+3}{n+1} \right) = \lim_{n\to\infty} (n+1) + \lim_{n\to\infty} \frac{2}{n+1} = +\infty + 0 = +\infty.$$

(LATER) ~~TH -3~~ BOLZANO - WEIERSTRASS THEOREM (APPEAR NEXT PAGE)

~~PROOF: (BOLZANO WEIERSTRASS THEOREM)~~

~~PROOF~~ EVERY BOUNDED SEQUENCE OF REAL NUMBERS HAS A CONVERGENT SUBSEQUENCE.

~~EXAMPLE~~: CONSIDER THE SEQUENCE $(a_n)_n$ GIVEN BY $a_n = (-1)^n + \frac{1}{n}$ THEN IT IS BOUNDED $|(-1)^n + \frac{1}{n}| \leq 1+1 = 2$.

BY BOLZANO WEIERSTRASS THEOREM, IT HAS A CONVERGENT SUBSEQUENCE.

WE CAN IDENTIFY THEM $a_{2k} = (-1)^{2k} + \frac{1}{2k} = 1 + \frac{1}{2k}$ AND $a_{2k+1} = (-1)^{2k+1} + \frac{1}{2k+1} = -1 + \frac{1}{2k+1}$

~~PROOF OF~~ BOLZANO - WEIERSTRASS LET $(a_n)_n$ BE A SEQUENCE OF REAL NUMBERS WHICH IS BOUNDED. WE HAVE SHOWN THAT IT ADMITS AN ACCUMULATION POINT $a \in \mathbb{R}$. THUS $\forall \varepsilon > 0$ $\forall N \in \mathbb{N}$ $\exists n \geq N$: $|a_n - a| < \varepsilon$

II – 3 – 5 MORE PROPERTIES

PROPOSITION: LET $(a_n)_n$ AND $(b_n)_n$ BE 2 CONVERGING SEQUENCES AND LET

$a := \lim_{n\to\infty} a_n$ AND $b := \lim_{n\to\infty} b_n$.

ASSUME $\exists N \in \mathbb{N}$ SUCH THAT $\forall n \geq N \quad b_n \geq a_n$.

THEN $a \leq b$.

PROOF: BY CONTRADICTION. ASSUME $a > b$. PUT $\varepsilon = \frac{a-b}{2} > 0$

BY CONVERGENCE, THERE EXIST $N_1, N_2 \in \mathbb{N}$:

$n \geq N_1 \Rightarrow |a_n - a| < \varepsilon$ AND $n \geq N_2 \Rightarrow |b_n - b| < \varepsilon$.

LET $n := \max\{N, N_1, N_2\}$. THEN $a_n > a - \varepsilon = \frac{a+b}{2}$

$a_n \in (a-\varepsilon, a+\varepsilon)$

AND $b_n < b + \varepsilon = \frac{a+b}{2}$. THUS $a_n > \frac{a+b}{2} > b_n$, A CONTRADICTION.

PROPOSITION: LET $(a_n)_n$ BE A CONVERGING SEQUENCE AND LET $a := \lim_{n\to\infty} a_n$.

THEN THE SEQUENCE $(|a_n|)_n$ CONVERGES AND $\lim_{n\to\infty} |a_n| = |a|$

(EXERCISE)

THEOREM ("SANDWICH THEOREM") LET $(a_n)_n$, $(b_n)_n$, $(c_n)_n$ BE SEQUENCES OF REAL NUMBERS. SUPPOSE THERE EXISTS $N_0 \in \mathbb{N}$ SUCH THAT

$a_n \leq b_n \leq c_n$ FOR EVERY $n \geq N_0$.

IF $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, THEN $\lim_{n\to\infty} b_n = L$

PROOF: LET $\varepsilon > 0$. SINCE $(a_n)_n$ AND $(c_n)_n$ CONVERGE TO L,

THERE EXIST $N_1, N_2 \in \mathbb{N}$ SUCH THAT $n \geq N_1 \Rightarrow a_n \in (L-\varepsilon, L+\varepsilon)$

$L - \varepsilon < a_n < L + \varepsilon$

AND $n \geq N_2 \quad L - \varepsilon < c_n < L + \varepsilon$.

SET $N = \max\{N_0, N_1, N_2\}$. FOR EVERY $n \geq N$

$L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon$, THUS $b_n \in (L-\varepsilon, L+\varepsilon)$

IT FOLLOWS $\lim_{n\to\infty} b_n = L$

EXAMPLE: $-\frac{1}{n} \le \frac{(-2)^n}{n} \le \frac{1}{n}$

$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{1}{n} = 0$ $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} -\frac{1}{n} = -\lim_{n\to\infty} \frac{1}{n} = 0$

$\Rightarrow \lim_{n\to\infty} b_n = 0$ ✓

PROPOSITION ("COMPARISON PRINCIPLE") LET $(a_n)_n$ AND $(b_n)_n$ SEQUENCES SUCH THAT $\exists N_0 \in \mathbb{N}$ WITH $a_n \le b_n \ \forall n \ge N_0$

1) ASSUME $\lim_{n\to\infty} a_n = +\infty$. THEN $\lim_{n\to\infty} b_n = +\infty$

2) ASSUME $\lim_{n\to\infty} b_n = -\infty$. THEN $\lim_{n\to\infty} a_n = -\infty$.

II ⇒ 5 UPPER AND LOWER LIMITS

LET $(a_n)_n$ BE BOUNDED SEQUENCE AND LET

$E := \{ x \in \mathbb{R} \mid x \text{ IS AN ACCUMULATION POINT} \}$

FACTS: • E IS NOT EMPTY, AS A BOUNDED SEQUENCE HAS AN ACCUMULATION POINT

• LET $M > 0$ BE SUCH THAT $|a_n| \le M$ FOR EVERY $n \in \mathbb{N}$.

IF $x \in E$, THERE EXISTS A SUBSEQUENCE $(b_k)_k = (a_{n_k})_k$ s.t. $\lim_k b_k = x$. AS $b_k = a_{n_k} \le M$ AND $b_k \ge -M$ WE OBTAIN $-M \le x \le M$. THUS $\exists \sup E$ AND $\exists \inf E$.

LET $(a_n)_n$ BE A BOUNDED SEQUENCE

DEFINITION: $\limsup_{n\to\infty} a_n := \sup E$ $\qquad \liminf_{n\to\infty} a_n := \inf E$.

FACT $\liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n$. $(a_n)_n$ CONVERGES $\Leftrightarrow \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$

IN THIS CASE $\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$

EXAMPLES: $a_n = (-1)^n + \frac{1}{n}$

$0, \frac{3}{2}, -\frac{2}{3}, \frac{5}{4}, -\frac{4}{5}, \frac{7}{6}$

EXAMPLE: $\lim_{n\to\infty} \frac{n^2+2n+3}{n+1}$ = BAD APPROACH $\lim_{n\to\infty} n^2+2n+3 = +\infty$

$\lim_{n\to\infty} n+1 = +\infty$, $\frac{+\infty}{+\infty}$ INDETERMINATE FORM.

CORRECT APPROACH:

$n^2+2n+3 = (n+1)^2 + 2$ WE OBTAIN $\frac{n^2+2n+3}{n+1} = \frac{(n+1)^2+2}{n+1} = n+1+\frac{2}{n+1}$

NOW $\lim_{n\to\infty}(n+1) = \lim_{n\to\infty} n + \lim_{n\to\infty} 1 = +\infty + 1 = +\infty$.

$\lim_{n\to\infty} \frac{2}{n+1} = \frac{\lim_{n\to\infty} 2}{\lim_{n\to\infty} n+1} = \frac{2}{+\infty} = 0$

$\Rightarrow \lim_{n\to\infty} \left(\frac{n^2+2n+3}{n+1}\right) = \lim_{n\to\infty}(n+1) + \lim_{n\to\infty}\frac{2}{n+1} = +\infty + 0 = +\infty$.

(LATER!) II -3- BOLZANO - WEIERSTRASS THEOREM (AFTER NEXT PAGE)

THEOREM (BOLZANO - WEIERSTRASS THEOREM)

EVERY BOUNDED SEQUENCE OF REAL NUMBERS HAS A CONVERGENT SUBSEQUENCE.

EXAMPLE: CONSIDER THE SEQUENCE $(a_n)_n$ GIVEN BY $a_n = (-1)^n + \frac{1}{n}$. THEN IT IS BOUNDED $|(-1)^n + \frac{1}{n}| \le 1+1 = 2$.

BY BOLZANO-WEIERSTRASS THEOREM, IT HAS A CONVERGENT SUBSEQUENCE. WE CAN IDENTIFY THEM $a_{2k} = (-1)^{2k} + \frac{1}{2k} = 1 + \frac{1}{2k}$ AND $a_{2k+1} = (-1)^{2k+1} + \frac{1}{2k+1} = -1 + \frac{1}{2k+1}$

PROOF OF BOLZANO-WEIERSTRASS: LET $(a_n)_n$ BE A SEQUENCE OF REAL NUMBERS WHICH IS BOUNDED. WE HAVE SHOWN THAT IT ADMITS AN ACCUMULATION POINT $a \in \mathbb{R}$. THUS $\forall \varepsilon > 0$ $\forall n \in \mathbb{N}$ $\exists n \ge N$: $|a_n - a| < \varepsilon$

WE CONSTRUCT A STRICTLY INCREASING SEQUENCE OF INDICES $n_1 < n_2 < \dots$

APPLYING THE PROPERTY WITH $\varepsilon = 1$ AND $N = 1$, WE CHOOSE $n_1 \geq 1$

SUCH THAT $|a_{n_1} - a| < 1$.

ONCE n_k HAS BEEN CHOSEN, APPLY THE PROPERTY WITH $\varepsilon = \frac{1}{k+1}$, $N = n_k + 1$

TO CHOOSE $n_{k+1} \geq n_k + 1$ SUCH THAT $|a_{n_{k+1}} - a| < \frac{1}{k+1}$

THE INDICES ARE STRICTLY INCREASING SO WE HAVE A SUBSEQUENCE

$b_k = a_{n_k}$ IS A SUBSEQUENCE OF a_n. BY CONSTRUCTION $|b_k - a| < \frac{1}{k}$

FORMALLY BEGIN. LET $\varepsilon > 0$, CHOOSE $K \in \mathbb{N}$ S.T. $K > 1/\varepsilon$

FOR EVERY $k \geq K$ WE HAVE THAT $|b_k - a| < \frac{1}{k} \leq \frac{1}{K} < \varepsilon$.

THEREFORE $\lim b_k = a$. WE HAVE FOUND A CONVERGING SUBSEQUENCE OF a_n

REMARK. THE PROOF SHOWS THAT EACH ACCUMULATION POINT IS REALIZED AS THE LIMIT OF A SUBSEQUENCE OF a_n

⚠ REMARK THE BOLZANO-WEIERSTRASS THEOREM ASSERTS CONVERGENCE OF *SOME* SUBSEQUENCE, NOT OF THE ORIGINAL SEQUENCE. $a_n = (-1)^n + \frac{1}{n}$ DOES NOT CONVERGE!

EXAMPLE: a_n b_n c_n

$$-\frac{1}{n} \le \frac{(-2)^n}{n} \le \frac{1}{n}$$

$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{1}{n} = 0$ $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} -\frac{1}{n} = -\lim_{n\to\infty} \frac{1}{n} = 0$

$\Rightarrow \lim_{n\to\infty} b_n = 0$!

PROPOSITION ("COMPARISON PRINCIPLE") LET $(a_n)_n$ AND $(b_n)_n$ SEQUENCES SUCH THAT $\exists N_0 \in \mathbb{N}$ WITH $a_n \le b_n \ \forall n \ge N_0$

1) ASSUME $\lim_{n\to\infty} a_n = +\infty$. THEN $\lim_{n\to\infty} b_n = +\infty$

2) ASSUME $\lim_{n\to\infty} b_n = -\infty$. THEN $\lim_{n\to\infty} a_n = -\infty$.

II 3 6 II - 3 - 6 UPPER AND LOWER LIMITS

LET $(a_n)_n$ BE BOUNDED SEQUENCE AND LET

$E := \{ x \in \mathbb{R} \mid x \text{ IS AN ACCUMULATION POINT} \}$

FACTS: • E IS NOT EMPTY, AS A BOUNDED SEQUENCE HAS AN ACCUMULATION POINT

• LET $M > 0$ BE SUCH THAT $|a_n| \le M$ FOR EVERY $n \in \mathbb{N}$.

IF $x \in E$, THERE EXISTS A SUBSEQUENCE $(b_k)_k = (a_{n_k})_k$ S.T. $\lim_{k\to\infty} b_k = x$. AS $b_k = a_{n_k} \le M$ AND $b_k \ge -M$ WE OBTAIN $-M \le x \le M$. THUS $\exists \sup E$ AND $\exists \inf E$.

LET $(a_n)_n$ IS A BOUNDED SEQUENCE

DEFINITION: $\lim_{n\to\infty} \sup a_n := \sup E$ $\quad \lim_{n\to\infty} \inf a_n = \inf E$.

FACT $\lim_{n\to\infty} \inf a_n \le \lim_{n\to\infty} \sup a_n$. ON CONVERGENCE ($\Leftarrow$) $\lim_{n\to\infty} \inf a_n = \lim_{n\to\infty} \sup a_n$ IN WHICH CASE $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \inf a_n = \lim_{n\to\infty} \sup a_n$

EXAMPLE: $a_n = (-1)^n + \frac{1}{n}$

$0, \frac{3}{2}, -\frac{2}{3}, \frac{5}{4}, -\frac{4}{5}, \frac{7}{6}$

$a_{2k} = 1 + \frac{1}{2k}$ $\quad \lim_{k\to\infty} a_{2k} = 1$

$a_{2k-1} = -1 + \frac{1}{2k-1}$ $\quad \lim_{k\to\infty} a_{2k-1} = -1$.

THERE ARE NO OTHER ACCUMULATION POINTS.

ASSUME x IS AN ACCUMULATION POINT WITH $x \neq -1, 1$.

PUT $\varepsilon = \frac{1}{3}\min\{|x-1|, |x+1|\} > 0$

CHOOSE $N \in \mathbb{N}$ SUCH THAT $1/N < \varepsilon$ FOR EVERY $n \geq N$

$a_n = (-1)^n + \frac{1}{n} \Rightarrow (-1)^n - x = (a_n - x) - \frac{1}{n}$, $|(-1)^n - x| \leq |a_n - x| + \frac{1}{n}$, $|(-1)^n - x| \geq 3\varepsilon$

$|a_n - x| \geq |(-1)^n - x| - \frac{1}{n} > 3\varepsilon - \varepsilon = 2\varepsilon$.

THUS NO TERM WITH $n \geq N$ BELONGS TO ε-NEIGHBOURHOOD OF x,

IN PARTICULAR x IS NOT AN ACCUMULATION POINT ! A CONTRADICTION

HENCE $E = \{-1, 1\}$. THUS $-1 = \liminf_{n\to\infty} a_n$

$1 = \limsup_{n\to\infty} a_n$

PROPOSITION LET (a_n) BE A BOUNDED SEQUENCE OF REAL NUMBERS - THEN

$L := \limsup a_n$ IS AN ACCUMULATION POINT OF (a_n). SAME FOR $\ell = \liminf a_n$

~~EXERCISE~~

PROOF: LET $\varepsilon > 0$ AND $N \in \mathbb{N}$. WE MUST FIND $n \geq N$ SUCH THAT $|a_n - L| < \varepsilon$

SINCE $L - \varepsilon/2 < L = \sup E$, THE NUMBER $L - \varepsilon/2$ IS NOT AN UPPER BOUND OF E. THUS THERE EXIST $x \in E$ SUCH THAT $L - \varepsilon/2 < x \leq L$

IN PARTICULAR $|x - L| < \varepsilon/2$

SINCE x IS AN ACCUMULATION POINT (THERE EXISTS $n \geq N$ SUCH THAT

$|a_n - x| < \varepsilon/2$

THEREFORE $|a_n - L| \leq |a_n - x| + |x - L| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$ HENCE L IS AN ACCUMULATION POINT. EXERCISE FOR $\liminf$.

PROPOSITION: LET (a_n) AND (b_n) BE BOUNDED SEQUENCES OF REAL NUMBERS.

1) ONE HAS THAT: $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup a_n + \limsup b_n$

2) ONE HAS THAT $\liminf (a_n + b_n) \geq \liminf a_n + \liminf b_n$

3) IF $(b_n)_n$ IS CONVERGENT THEN $\limsup_{n\to\infty}(a_n+b_n) = \limsup_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.

$$\liminf_{n\to\infty}(a_n+b_n) = \liminf_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

4) $$\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n$$

$$\liminf_{n\to\infty}(-a_n) = -\limsup_{n\to\infty} a_n$$

~~PROOF~~

△ REMARK IT IS FALSE (IN GENERAL) THAT $\limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n = \limsup_{n\to\infty}(a_n+b_n)$

EXAMPLE $a_n = (-1)^n$, $b_n = (-1)^{n+1}$ $\quad (-1)^n + (-1)^{n+1} = 0.$

$\limsup_{n\to\infty} a_n = 1$ $\quad \limsup_{n\to\infty} b_n = 1$ $\quad \limsup_{n\to\infty}|a_n+b_n| = \limsup_{n\to\infty} 0 = 0$

~~BOLZANO-WEIERSTRASS~~

PROOF: 1) LET $A := \limsup_{n\to\infty} a_n$, $B := \limsup_{n\to\infty} b_n$, $C := \limsup_{n\to\infty}(a_n+b_n)$

~~SINCE a_n+b_n IS BOUNDED (AS A SUM OF TWO BOUNDED SEQUENCES)~~

C IS AN ACCUMULATION POINT OF a_n+b_n. THERE EXISTS A SUBSEQUENCE $(a_{n_k}+b_{n_k})_k$ SUCH THAT $\lim_{k\to\infty}(a_{n_k}+b_{n_k}) = C$

THE SEQUENCE $(a_{n_k})_k$ IS BOUNDED. BY THE BOLZANO-WEIERSTRASS THEOREM, IT HAS A CONVERGENT SUBSEQUENCE.

$$\lim_{j\to\infty} a_{n_{k_j}} = L \qquad (k_j)_j \to (n_{k_j})_j$$

SINCE L IS AN ACCUMULATION POINT, WE HAVE $L \le A$

NOW $\lim_{j\to\infty} b_{n_{k_j}} = \lim_{j\to\infty}\left((a_{n_{k_j}}+b_{n_{k_j}}) - a_{n_{k_j}}\right) = C-L$. THUS $C-L$ IS AN ACCUMULATION POINT. WE DEDUCE $C-L \le B$.

HENCE $C \le L+B \le A+B$ ∎

LET $b := \lim_{n\to\infty} b_n$

3) WE HAVE THAT $\limsup_{n\to\infty}(a_n+b_n) \le \limsup_{n\to\infty}(a_n) + \limsup_{n\to\infty}(b_n)$

$= \limsup_{n\to\infty}(a_n) + \lim_{n\to\infty} b_n$

$= \limsup_{n\to\infty} a_n + b$

WE HAVE THAT $a_n = (a_n+b_n) + (-b_n)$

WE HAVE $\lim_{n\to\infty}(-b_n) = -b$. APPLYING 1) AGAIN YIELDS

$\limsup_{n\to\infty} a_n \le \limsup_{n\to\infty}(a_n+b_n) - b$ FROM WHENCE $\limsup_{n\to\infty} a_n + b \le \limsup_{n\to\infty}(a_n+b_n)$

NOW $\limsup_{n\to\infty} a_n + b \le \limsup_{n\to\infty}(a_n+b_n) \le \limsup_{n\to\infty} a_n + b \Rightarrow \limsup_{n\to\infty}(a_n+b_n) = \limsup_{n\to\infty} a_n + b$

PROPOSITION ASSUME $(a_n)_n$ AND $(b_n)_n$ ARE BOUNDED SEQUENCES OF NON-NEGATIVE REAL NUMBERS.

THEN $(\liminf_{n\to\infty} a_n)(\liminf_{n\to\infty} b_n) \le \liminf_{n\to\infty} a_n b_n \le \limsup_{n\to\infty} a_n b_n$

$\le (\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n)$

IF IN ADDITION $\lim_{n\to\infty} b_n = b$ THEN $\liminf_{n\to\infty} a_n b_n = \liminf_{n\to\infty} a_n \cdot b$

$\limsup_{n\to\infty} a_n b_n = \limsup_{n\to\infty} a_n \cdot b$

RECIPROCALS ASSUME $\exists\, m, M > 0 : \forall n \quad m \le a_n \le M$

THEN $\limsup_{n\to\infty} \frac{1}{a_n} = \frac{1}{\liminf_{n\to\infty} a_n}$

$\liminf_{n\to\infty} \frac{1}{a_n} = \frac{1}{\limsup_{n\to\infty} a_n}$

④

PROOF: (RECIPROCALS) ASS. T $\alpha := \liminf_{n\to\infty} a_n$. WE SHOW THAT $1/\alpha$ IS THE LARGEST ACCUMULATION POINT OF $\left(\frac{1}{a_n}\right)_n$. FIRST, $1/\alpha$ IS AN ACCUMULATION POINT:

IF $b_k := (a_{n_k})_k$ IS A SUBSEQUENCE CONVERGING TO α, THEN $\left(\frac{1}{b_k}\right)_k$ IS A SUBSEQUENCE OF $\left(\frac{1}{a_n}\right)_n$ CONVERGING TO $1/\alpha$. THEN y BE ANOTHER ACCUMULATION POINT OF $\left(\frac{1}{a_n}\right)_n$

CHOOSE A SUBSEQUENCE $(c_k)_k$ OF $\left(\frac{1}{a_n}\right)_n$ CONVERGING TO y. THEN $\left(\frac{1}{c_k}\right)_k$ IS A SUBSEQUENCE OF $(a_n)_n$ CONVERGING TO $\frac{1}{y}$. IT FOLLOWS $\frac{1}{y} \ge \alpha$, THAT IS $\frac{1}{\alpha} \ge y$. HENCE $\frac{1}{\alpha}$ IS THE LARGEST ACCUMULATION POINT OF $\left(\frac{1}{a_n}\right)_n$

(LATEX)

II-4 CAUCHY SEQUENCES (AFTER PAGE PREVIOUS)

II-4-1 - CAUCHY SEQUENCES

DEFINITION: A SEQUENCE $(a_n)_{n\in\mathbb{N}}$ OF REAL NUMBERS IS CALLED A CAUCHY SEQUENCE IF $\forall \varepsilon>0 \ \exists N\in\mathbb{N} \ \forall m,n\geq N : |a_m - a_n| < \varepsilon$

"THE CAUCHY CONDITION"

EXAMPLE: THE SEQUENCE $a_n = \frac{1}{n}$ IS A CAUCHY SEQUENCE.

PROOF: LET $\varepsilon>0$. CHOOSE $N > 1/\varepsilon$ (BY ARCHIMEDEAN PROPERTY)

LET $m,n \geq N$ THEN $0<\frac{1}{m}\leq\frac{1}{N}$ AND $0<\frac{1}{n}\leq\frac{1}{N}$.

WE HAVE THAT $\left|\frac{1}{m}-\frac{1}{n}\right| \leq$

$$\left|\frac{1}{N}-0\right| = \frac{1}{N} < \varepsilon.$$

NON-EXAMPLE: THE SEQUENCE $a_n=(-1)^n$ IS NOT A CAUCHY SEQUENCE.

INDEED, TAKE $\varepsilon=1$. FOR EVERY $N\in\mathbb{N}$, CHOOSING $m=N$ AND $n=N+1$

GIVES: IF N IS EVEN $a_N = 1$ $a_{N+1}=-1$

$$|a_N - a_m| = |a_N - a_{N+1}| = |1-(-1)| = 2$$

IF N IS ODD, $a_N=-1$, $a_{N+1}=1$

$$|a_n - a_m| = |a_N - a_{N+1}| = |(-1)-1| = 2.$$

THUS FOR EVERY $N\in\mathbb{N}$ THERE EXISTS INDICES $m,n\geq N$ s.t. $|a_n-a_m| \geq 1$,

THUS a_n IS NOT CAUCHY.

(2) (AFTER NEXT PAGE)

PROPOSITION: EVERY CAUCHY SEQUENCE IS BOUNDED.

PROOF: LET $(a_n)_n$ BE A CAUCHY SEQUENCE. APPLYING THE CAUCHY CONDITION WITH $\varepsilon=1$, THERE EXISTS $N\in\mathbb{N}$ SUCH THAT $|a_n-a_m|<1$ FOR ALL $m,n\geq N$. TAKING $m=N$ WE OBTAIN THAT FOR EVERY $n\geq N$

$$|a_n| \leq |a_n - a_N| + |a_N| < 1 + |a_N|$$

SET $C := \max\{|a_1|, \dots, |a_N|, 1+|a_N|\}$. FOR EVERY $n\in\mathbb{N}$

WE HAVE THAT $|a_n|\leq C$, HENCE $(a_n)_n$ IS BOUNDED.