Analysis I
UP - FAMNIT
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Tutorial 6

1. Consider the following sequences of intervals, defined for $n \geq 1$:

   (a) $I_n = \left[2 - \frac{1}{n},\, 2 + \frac{1}{n}\right]$,
   (b) $I_n = \left[0,\, 1 + \frac{1}{n}\right]$,
   (c) $I_n = \left[\frac{1}{n},\, 2 - \frac{1}{n}\right]$,
   (d) $I_n = [n, \infty)$.

   For each sequence:

   i. Determine whether the intervals are nested in the sense that $I_{n+1} \subseteq I_n$ for every $n \geq 1$.
   ii. State whether the Nested Intervals Theorem applies. Justify your answer.
   iii. Determine $\bigcap_{n=1}^{\infty} I_n$ and justify your answer.

2. Use arithmetic of limits (AoL) to prove that $s_n \to \frac{1}{5}$ as $n \to \infty$, where

$$s_n = \frac{n^3 + 6n^2 + 7}{5n^3 + 3n - 4}.$$

3. For each $a \in \mathbb{R}$, give an example of a convergent sequence (s_n) such that $s_n > a$ for all n, but

$$\lim_{n\to\infty} s_n = a.$$

4. By using the limit definition, prove that $\lim_{n\to\infty} (\sqrt{n} + 7) = +\infty$.

5. Find the following limits by using the "Arithmetic of Limits".

   (i) $\lim_{n\to\infty} \dfrac{3n^2 - 5n}{8n^2 + 4n - 1}$.

   (ii) $\lim_{n\to\infty} \dfrac{\sqrt{5n^2 + n}}{2n + 3}$.

   (iii) $\lim_{n\to\infty} \dfrac{-6n^3 + 1}{2n^3 + 2n^2 - n}$.

   (iv) $\lim_{n\to\infty} \dfrac{n^2 - 3}{2n^3 + 2n + 3}$.

   (v) $\lim_{n\to\infty} \dfrac{\sqrt{3n^3 - n}}{3n - 1}$.

   (vi) $\lim_{n\to\infty} \dfrac{\sqrt[3]{2n^2 - n^4}}{2n^3 + 3n^2 + 5n}$.

6. Consider the sequence $a_n = \frac{\sin n}{n}$, $n \geq 1$. Use the Sandwich Theorem to prove that (a_n) converges.

7. Consider the sequence
$$b_n = \frac{n \cos n}{n^2 + 1}, \qquad n \geq 1.$$
   (a) Find two sequences (u_n) and (v_n) such that
   $$u_n \leq b_n \leq v_n \qquad \text{for every } n \geq 1.$$
   (b) Show that (u_n) and (v_n) converge to the same limit.
   (c) Apply the Sandwich Theorem to determine $\lim_{n\to\infty} b_n$.

8. Prove that the sequence defined by its nth term
$$\frac{(n^2 + 1) \sin n^3}{n^2 + n}$$
   has a convergent subsequence.

9. Consider the sequence
$$a_n = (-1)^n + \frac{1}{n}, \qquad n \geq 1.$$
   (a) Show that (a_n) is bounded.
   (b) What does the Bolzano–Weierstrass Theorem tell us about this sequence?
   (c) Find two convergent subsequences of (a_n) and determine their limits.
   (d) Does (a_n) converge? Justify your answer.

Some Solutions

1. **Theorem (Nested Intervals).** If I_n are nonempty, closed, bounded intervals in $\mathbb{R}$ and $I_{n+1} \subseteq I_n$ for all $n \geq 1$, then

$$\bigcap_{n=1}^{\infty} I_n \neq \varnothing.$$

(a) $I_n = \left[2 - \frac{1}{n},\, 2 + \frac{1}{n}\right]$.

Are they nested? Yes. Since $\frac{1}{n+1} < \frac{1}{n}$, we have

$$2 - \frac{1}{n} < 2 - \frac{1}{n+1} \quad \text{and} \quad 2 + \frac{1}{n+1} < 2 + \frac{1}{n}.$$

Thus both endpoints move inward, so $I_{n+1} \subseteq I_n$.

Does the theorem apply? Yes. Each interval is nonempty, closed, and bounded, and the intervals are nested. Therefore the intersection is nonempty.

What is the intersection? Since $2 \in I_n$ for every $n \in \mathbb{N}$ and

$$2 - \frac{1}{n} \leq x \leq 2 + \frac{1}{n}$$

for every $x \in \bigcap_{n=1}^{\infty} I_n$, letting $n \to \infty$ gives $x = 2$. Hence,

$$\bigcap_{n=1}^{\infty} I_n = \{2\}.$$

(b) $I_n = \left[0,\, 1 + \frac{1}{n}\right]$.

Are they nested? Yes. The left endpoint remains 0, and

$$1 + \frac{1}{n+1} < 1 + \frac{1}{n} \quad \Longrightarrow \quad I_{n+1} \subseteq I_n.$$

Does the theorem apply? Yes. Every interval is nonempty, closed, and bounded, and the intervals are nested.

What is the intersection? Every $x \in [0, 1]$ belongs to every I_n, since $0 \leq x \leq 1 < 1 + \frac{1}{n}$. Conversely, if x belongs to every I_n, then $0 \leq x \leq 1 + \frac{1}{n}$ for all $n \in \mathbb{N}$. Letting $n \to \infty$, we obtain $0 \leq x \leq 1$. Hence,

$$\bigcap_{n=1}^{\infty} I_n = [0, 1].$$

(c) $I_n = \left[\frac{1}{n},\, 2 - \frac{1}{n}\right]$.

Are they nested? Not in the direction required by the theorem. The first intervals are

$$I_1 = [1,1] = \{1\}, \qquad I_2 = \left[\frac{1}{2}, \frac{3}{2}\right], \qquad I_3 = \left[\frac{1}{3}, \frac{5}{3}\right].$$

Since $1/(n+1) < 1/n$ and $2 - 1/(n+1) > 2 - 1/n$, we obtain

$$I_n \subseteq I_{n+1} \qquad \text{for every } n.$$

These intervals expand rather than shrink; in particular, $I_2 \nsubseteq I_1$.

Does the theorem apply? No, because the required inclusion $I_{n+1} \subseteq I_n$ fails. The intervals are nonempty, closed, and bounded, but this alone is not enough to invoke the theorem.

What is the intersection? Since $I_1 = \{1\}$, any point in the intersection must be 1. Moreover, for every $n \geq 1$,

$$\frac{1}{n} \leq 1 \leq 2 - \frac{1}{n},$$

so 1 belongs to every interval. Therefore

$$\bigcap_{n=1}^{\infty} I_n = \{1\}.$$

(d) $I_n = [n, \infty)$.

Are they nested? Yes. If $x \in I_{n+1}$, then $x \geq n+1 > n$, so $x \in I_n$. Hence $I_{n+1} \subseteq I_n$.

Does the theorem apply? No. Although each interval is nonempty and closed and the sequence is nested, the intervals are unbounded.

What is the intersection? If x belonged to every I_n, then $x \geq n$ for every natural number n. By the Archimedean property, we can choose $n > x$, which is impossible. Therefore

$$\bigcap_{n=1}^{\infty} I_n = \varnothing.$$

Summary.

Part	$I_{n+1} \subseteq I_n$?	Theorem applies?	Intersection
(a)	Yes	Yes	$\{2\}$
(b)	Yes	Yes	$[0,1]$
(c)	No	No	$\{1\}$
(d)	Yes	No	$\varnothing$

Definition (Convergence). A sequence (a_n) converges to $L \in \mathbb{R}$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies |a_n - L| < \varepsilon.$$

Definition (Divergence to $+\infty$). We say that $a_n \to +\infty$ if for every $M > 0$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies a_n > M.$$

4. We must show that for every $M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \implies \quad \sqrt{n} + 7 > M.$$

Let $M \in \mathbb{R}$ and choose $N \in \mathbb{N}$ such that $N > M^2$. If $n \geq N$, then

$$\sqrt{n} + 7 \geq \sqrt{N} + 7 > M + 7 \geq M.$$

Thus the defining condition holds for every real M, and hence

$$\lim_{n\to\infty} (\sqrt{n} + 7) = +\infty.$$

5. (i) $\lim_{n\to\infty} \frac{3-\frac{5}{n}}{8+\frac{4}{n}-\frac{1}{n^2}} = \frac{3}{8}$.

(ii) $\lim_{n\to\infty} \frac{\sqrt{5+\frac{1}{n}}}{2+\frac{3}{n}} = \frac{\sqrt{5}}{2}$.

(iii) $\lim_{n\to\infty} \frac{-6+\frac{1}{n^3}}{2+\frac{2}{n}-\frac{1}{n^2}} = -3$.

(iv) $\lim_{n\to\infty} \frac{\frac{1}{n}-\frac{3}{n^3}}{2+\frac{2}{n^2}+\frac{3}{n^3}} = 0$.

(v) $\lim_{n\to\infty} \sqrt{n}\, \frac{\sqrt{3-\frac{1}{n^2}}}{3-\frac{1}{n}} = +\infty$.

(vi) $\lim_{n\to\infty} \frac{\sqrt[3]{\frac{2}{n^2}-1}}{n^{5/3}\left(2+\frac{3}{n}+\frac{5}{n^2}\right)} = 0$.

7. We are given

$$b_n = \frac{n \cos n}{n^2 + 1}, \qquad n \geq 1.$$

(a) Since $-1 \leq \cos n \leq 1$ and $n/(n^2+1) > 0$, multiplying gives

$$-\frac{n}{n^2+1} \leq \frac{n \cos n}{n^2+1} \leq \frac{n}{n^2+1}.$$

Thus we may take

$$u_n = -\frac{n}{n^2+1}, \qquad v_n = \frac{n}{n^2+1}.$$

(b) For every $n \geq 1$,

$$0 \leq v_n = \frac{n}{n^2+1} \leq \frac{1}{n}.$$

Since $1/n \to 0$, the Sandwich Theorem gives $v_n \to 0$. Consequently, $u_n = -v_n \to 0$ as well.

(c) We have $u_n \leq b_n \leq v_n$ and both bounds converge to 0. Hence, by the Sandwich Theorem,

$$\lim_{n\to\infty} \frac{n \cos n}{n^2+1} = 0.$$

9. We are given
$$a_n = (-1)^n + \frac{1}{n}, \qquad n \geq 1.$$
(a) We know that $-1 \leq (-1)^n \leq 1$ and $0 < 1/n \leq 1$. Therefore
$$-1 < a_n \leq 2 \qquad \text{for every } n \geq 1,$$
so (a_n) is bounded.
(b) The Bolzano–Weierstrass Theorem states that every bounded sequence of real numbers has a convergent subsequence. Hence (a_n) has at least one convergent subsequence.
(c) For even indices,
$$a_{2k} = 1 + \frac{1}{2k} \longrightarrow 1 \qquad (k \to \infty).$$
For odd indices,
$$a_{2k-1} = -1 + \frac{1}{2k-1} \longrightarrow -1 \qquad (k \to \infty).$$
Thus we have found two convergent subsequences with distinct limits, 1 and -1.
(d) Suppose, to the contrary, that $a_n \to L$. Then every subsequence of (a_n) must also converge to L. However, the even-indexed subsequence converges to 1, whereas the odd-indexed subsequence converges to -1. By uniqueness of limits this is impossible. Therefore (a_n) does *not* converge.