MATHEMATICAL INDUCTION

(FIRST)

Principle of Mathematical Induction

IF S IS A SET OF INTEGERS SUCH THAT

(i) $a \in S$

(ii) FOR ALL $k \geq a$, IF $k \in S$ THEN $k+1 \in S$

THEN $S = \{ m \in \mathbb{Z} : m \geq a \}$.

WE USUALLY USE MATHEMATICAL INDUCTION FOR ATTEMPTING TO PROVE A STATEMENT ABOUT THE POSITIVE INTEGERS. WHEN GIVING INDUCTION PROOFS, WE OFTEN SHORTEN THE ARGUMENTS BY ELIMINATING ALL REFERENCE TO THE SET S. IT IS ALSO IMPORTANT TO ESTABLISH BOTH CONDITIONS (i)-(ii) BEFORE GIVING A CONCLUSION. THE PROOF OF CONDITION (i) IS USUALLY CALLED THE BASIS FOR THE INDUCTION, WHILE THE PROOF OF (ii) IS CALLED THE INDUCTION STEP. THE ASSUMPTIONS MADE IN THE INDUCTION STEP ARE KNOWN AS THE INDUCTION HYPOTHESES.

EXERCISE: PROVE BY MATHEMATICAL INDUCTION THAT

$$1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{n\cdot(2n-1)(2n+1)}{3}$$

FOR ALL $n \geq 1$.

PROOF: WE FIRST ABBREVIATE

$$1^2 + 3^2 + \dots + (2n-1)^2 = \sum_{k=1}^{n} (2k-1)^2 .$$

WE NEXT CONSIDER THE SET S DEFINED AS FOLLOWS:

$$S = \left\{ n \in \mathbb{N} : \sum_{k=1}^{n} (2k-1)^2 = \frac{n\cdot(2n-1)\cdot(2n+1)}{3} \right\}$$

BY CONSTRUCTION $S \subseteq \mathbb{N}$. WE WOULD LIKE TO PROVE THAT $S = \mathbb{N}$. TO DO THIS, WE WILL PROCEED BY USING THE FIRST PRINCIPLE OF MATHEMATICAL INDUCTION.

(i) BASIS FOR THE INDUCTION: WE NEED TO CHECK THAT $1 \in S$. BY THE DEFINITION OF S, THIS IS EQUIVALENT TO CHECK THE FOLLOWING EQUALITY HOLDS:

$$\sum_{k=1}^{1} (2k-1)^2 = \frac{1\cdot(2\cdot1-1)\cdot(2\cdot1+1)}{3} . \qquad (*)$$

LHS: $\sum_{k=1}^{1}(2k-1)^2 = (2\cdot 1-1)^2 = (2-1)^2 = 1^2 = 1.$

RHS: $\frac{1\cdot(2\cdot 1-1)\cdot(2\cdot 1+1)}{3} = \frac{1\cdot(2-1)\cdot(2+1)}{3} = \frac{3}{3} = 1.$

SINCE THE LHS OF (*) EQUALS THE RHS OF (*) WE THEREFORE HAVE $1 \in S$.

(ii) INDUCTION STEP: LET n BE AN ARBITRARY POSITIVE INTEGER $n > 1$. ASSUME THAT $n \in S$. THIS MEANS, BY THE DEFINITION OF S,

$$\sum_{k=1}^{n}(2k-1)^2 = \frac{n\cdot(2n-1)\cdot(2n+1)}{3}$$

(INDUCTION HYPOTHESES)

WE NEXT SHOW THAT $n+1 \in S$. NOTE THAT

$$\begin{aligned}
\sum_{k=1}^{n+1}(2k-1)^2 &= 1^2+3^2+\ldots+(2n-1)^2+[2(n+1)-1]^2 \\
&= \sum_{k=1}^{n}(2k-1)^2 + [2(n+1)-1]^2 \\
&= \frac{n\cdot(2n-1)\cdot(2n+1)}{3} + (2n+1)^2 \\
&= (2n+1)\cdot\left[\frac{n(2n-1)}{3} + (2n+1)\right]
\end{aligned}$$

BY THE INDUCTION HYPOTHESES

$(n+1)(2(n+1)-1)(2(n+1)+1)$ $\quad (n+1)(2n+3)$

$$\frac{(n+1)(2n+1)(2n+3)}{3} = \underline{(2n+1)}\left[\frac{n(2n-1)+3\cdot(2n+1)}{3}\right].$$

WE NOW OBSERVE THE FOLLOWING HOLD:

$$\underbrace{n(2n-1)+3(2n+1)}_{\substack{2n^2-n+6n+3 \\ 2n^2+5n+3=0}} = n(n+n-1)+3(n+n+1)$$

$$= n(n\underline{+1}+n-1\underline{-1})+3(n+1)+3n$$

$$= n(n+1)+n(n-2)+3(n+1)+3n$$

$$= (n+1)\cdot(n+3)+n(n-2+3)$$

$$= (n+1)\cdot(n+3)+n(n+1)$$

$$= (n+1)(n+3+n)$$

$$= (n+1)(2n+3)$$

$$= (n+1)(2(n+1)+1)$$

Bhaskara formula

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a}=x_{1,2}$$

$$2n^2+5n+3 = 2(n-x_1)(n-x_2)$$

IT FOLLOWS FROM THE ABOVE COMMENTS THAT

$$\sum_{k=1}^{n+1}(2k-1)^2 = \frac{(2n+1)(n+1)(2(n+1)+1)}{3}$$

$$= \frac{(n+1)(2(n+1)-1)(2(n+1)+1)}{3}.$$

BY THE DEFINITION OF S, THIS SHOWS THAT

$n+1 \in S$ WHENEVER $n \in S$.

THEREFORE, AS CONDITIONS (i)–(ii) HOLD, BY

THE FIRST PRINCIPLE OF MATHEMATICAL INDUCTION

WE HAVE $S = \mathbb{N}$. IN THIS CASE, THIS MEANS

$$\sum_{k=1}^{n} (2k-1)^2 = \frac{n \cdot (2n-1) \cdot (2n+1)}{3} \quad \forall n \geq 1$$

THIS CONCLUDES OUR PROOF. $\blacksquare$

EXERCISE: PROVE BY MATHEMATICAL INDUCTION THAT

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n \cdot (n+1)}{2}\right)^2$$

FOR ALL $n \geq 1$.

PROOF: WE FIRST ABBREVIATE

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \sum_{k=1}^{n} k^3 .$$

SO, WE NEED TO PROVE FOR ALL $n \geq 1$,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n \cdot (n+1)}{2}\right)^2 . \qquad (*)$$

WE WILL PROCEED BY MATHEMATICAL INDUCTION.

(i) BASIS OF THE INDUCTION: WE NEED TO CHECK THAT (*) HOLDS WHEN $n=1$.

LHS: $\sum_{k=1}^{1} k^3 = 1^3 = 1$.

RHS: $\left(\frac{1\cdot(1+1)}{2}\right)^2 = \left(\frac{1\cdot 2}{2}\right)^2 = 1^2 = 1$.

WE THUS HAVE (*) HOLDS WHEN $n=1$.

(ii) INDUCTION STEP: LET $h \in \mathbb{N}$, $h > 1$. ASS ASSUME BY INDUCTION HYPOTHESES THAT (*) HOLDS WHEN $n=h$. THAT'S IS,

$$\sum_{k=1}^{h} k^3 = \left(\frac{h\cdot(h+1)}{2}\right)^2$$

(Induction Hypotheses)

WE NEXT ATTEMPT TO PROVE THE DESIRED EQUALITY FOR $h+1$:

$$\sum_{k=1}^{h+1} k^3 = 1^3 + 2^3 + 3^3 + \dots + h^3 + (h+1)^3$$

$$= \sum_{k=1}^{h} k^3 + (h+1)^3$$

$$= \left(\frac{h\cdot(h+1)}{2}\right)^2 + (h+1)^3$$

(Induction Hypotheses)

$$= (n+1)^2 \cdot \left[\frac{n^2}{2^2} + (n+1) \right]$$

$$= (n+1)^2 \cdot \frac{n^2 + 4(n+1)}{4}$$

$$= \frac{(n+1)^2 \ (n^2 + 4n + 4)}{4}$$

$$= \left[\frac{(n+1)\ (n+2)}{2} \right]^2 = \left[\frac{(n+1)((n+1)+1)}{2} \right]^2$$

AS WE WANTED.

ACCORDING TO THE PRINCIPLE OF INDUCTION, SINCE (i) – (ii) HOLD, WE HAVE THE GIVEN EQUALITY IS VALID FOR ALL $n \geq 1$.

THIS CONCLUDES OUR PROOF. ■

EXERCISE: PROVE THAT THE CUBE OF ANY INTEGER CAN BE WRITTEN AS THE DIFFERENCE OF TWO SQUARES. $m^3 = p^2 - q^2$

PROOF: WE FIRST NOTICE FOR ALL $m \in \mathbb{N}$,

$$m^3 = \cancel{1^3} + \cancel{2^3} + \cancel{\cdots} + \cancel{(m-1)^3} + m^3 - \cancel{1^3} - \cancel{2^3} - \cancel{\cdots} - \cancel{(m-1)^3}$$

$$\underline{m^3} = \left(1^3+2^3+\dots+(m-1)^3+m^3\right) - \left(1^3+2^3+\dots+(m-1)^3\right)$$

WE WILL NOW APPLY WHAT WE PROVED IN A PREVIOUS EXERCISE: WE KNOW FOR EVERY $p \in \mathbb{N}$,

$$\sum_{k=1}^{p} k^3 = \left(\frac{p\cdot(p+1)}{2}\right)^2.$$

$p = m$

$p = m-1$

IN PARTICULAR, WE HAVE

$$\sum_{k=1}^{m} k^3 = \left(\frac{m(m+1)}{2}\right)^2, \qquad \sum_{k=1}^{m-1} k^3 = \left[\frac{(m-1)\cdot m}{2}\right]^2.$$

THEREFORE, FROM THE ABOVE COMMENTS,

$$m^3 = \left(\frac{m(m+1)}{2}\right)^2 - \left(\frac{m(m-1)}{2}\right)^2.$$

WE NOW NOTICE $\frac{m}{2}$ IS AN INTEGER IF m IS EVEN WHILE $\frac{m+1}{2}$ AND $\frac{m-1}{2}$ ARE INTEGERS IF m IS ODD.

HENCE, FOR ALL $m \in \mathbb{N}$, THE NUMBERS

$\frac{m(m+1)}{2}$ AND $\frac{m(m-1)}{2}$ ARE INTEGERS.

THUS, m^3 IS THE DIFFERENCE OF TWO SQUARES. $\blacksquare$

(SECOND)

Principle of Strong Mathematical Induction

IF S IS A SET OF INTEGERS SUCH THAT

(i) $a \in S$ ✓

(ii) IF EVERY INTEGER k WITH $a \leq k \leq m$ BELONGS TO S THEN $m+1 \in S$

$a \in S$ ✓
$a+1 \in S$ ✓
$a+2 \in S$
$\vdots$
$m \in S$ ✓

IND HYPOTHESIS

$m+1 \in S$

THEN, $S = \{ m \in \mathbb{N} : m \geq a \}$.

EXAMPLE: LET a BE A POSITIVE REAL NUMBER SUCH THAT $a + \frac{1}{a}$ IS AN INTEGER.

PROVE THAT $a^m + \frac{1}{a^m} \in \mathbb{Z}$ FOR ALL $m \geq 1$.

PROOF: LET a BE A POSITIVE REAL NUMBER SUCH THAT $a + a^{-1} \in \mathbb{Z}$. WE WILL PROCEED BY INDUCTION ON m. NOTICE

$$\left(a + \frac{1}{a}\right)^2 = a^2 + 2 \cdot \cancel{a} \cdot \frac{1}{\cancel{a}} + \frac{1}{a^2} = a^2 + \frac{1}{a^2} + 2$$

THEN, WE CAN WRITE

$$a^2 + \frac{1}{a^2} = \overbrace{\underbrace{\left(a + \frac{1}{a}\right)}_{\in \mathbb{Z}}^2 - 2}^{\in \mathbb{Z}} \in \mathbb{Z}$$

AS $a + \frac{1}{a} \in \mathbb{Z}$ WE HAVE $a^2 + \frac{1}{a^2} \in \mathbb{Z}$.

THIS PROVES THE BASE CASE OF THE INDUCTION.

ASSUME NEXT THAT $a^k + \frac{1}{a^k} \in \mathbb{Z}$ FOR ALL INTEGERS $1 \le k \le m$. WE WILL ATTEMPT TO PROVE $a^{m+1} + \frac{1}{a^{m+1}} \in \mathbb{Z}$ AS WELL. WE HAVE

$$\Big(\underbrace{a^m + \frac{1}{a^m}}_{\in \mathbb{Z}}\Big)\Big(\underbrace{a + \frac{1}{a}}_{\in \mathbb{Z}}\Big) = a^{m+1} + a^m \cdot \frac{1}{a} + a \cdot \frac{1}{a^m} + \frac{1}{a^{m+1}}$$

$$= a^{m+1} + a^{m-1} + \frac{1}{a^{m-1}} + \frac{1}{a^{m+1}}$$

THEN, WE CAN WRITE

$k=m$ $\quad$ $k=1$ $\quad$ $k=m-1$

$$a^{m+1} + \frac{1}{a^{m+1}} = \underbrace{\Big(a^m + \frac{1}{a^m}\Big)}_{\in \mathbb{Z}} \cdot \underbrace{\Big(a + \frac{1}{a}\Big)}_{\in \mathbb{Z}} - \underbrace{\Big(a^{m-1} + \frac{1}{a^{m-1}}\Big)}_{\in \mathbb{Z}} \in \mathbb{Z}$$

FROM THE INDUCTIVE HYPOTHESIS, THE NUMBERS

$a^m + \frac{1}{a^m}$, $a + \frac{1}{a}$, $a^{m-1} + \frac{1}{a^{m-1}}$ ARE ALL INTEGERS.

THIS IMPLIES THAT $a^{m+1} + \frac{1}{a^{m+1}} \in \mathbb{Z}$.

THEREFORE, BY THE PRINCIPLE OF STRONG MATHEMATICAL INDUCTION, THE NUMBER $a^m + \frac{1}{a^m}$ IS AN INTEGER FOR ALL $m \ge 1$. ∎

$a_4 = a_3 + a_2 + a_1 = 3 + 2 + 1 = 6$ $\qquad$ $a_5 < 2^5 = 32$ ✓

$a_5 = a_4 + a_3 + a_2 = 6+3+2 = 11$ $\qquad a_4 < 2^4 = 16$ ✓

EXAMPLE: SUPPOSE THAT THE NUMBERS a_m ARE DEFINED INDUCTIVELY BY $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ AND $a_m = a_{m-1} + a_{m-2} + a_{m-3}$ FOR ALL $m \geq 4$. PROVE THAT $a_m < 2^m$ FOR EVERY $m \in \mathbb{N}$.

PROOF: WE FIRST NOTE $a_m < 2^m$ FOR EVERY $m \in \{1, 2, 3\}$. IN FACT, THE INEQUALITIES $1 < 2$, $2 < 2^2 = 4$, $3 < 2^3 = 8$ ARE ALL TRUE.

$1 < 2^1 = 2$
$2 < 2^2 = 4$
$3 < 2^3 = 8$

LET $m \geq 4$ AND ASSUME $a_k < 2^k$ FOR ALL $4 \leq k \leq m$.
WE WILL NEXT SHOW THAT $a_{m+1} < 2^{m+1}$.
BY DEFINITION WE HAVE

$$a_{m+1} = a_m + a_{m-1} + a_{m-2}$$

WE NEXT CONSIDER SOME CASES:

(i) IF $m = 4$ WE HAVE $a_5 = a_4 + a_3 + a_2$.
BY INDUCTION HYPOTHESIS, $a_4 < 2^4$. RECALL $a_2 = 2$, $a_3 = 3$. WE THEREFORE HAVE:

$$a_5 = a_4 + a_3 + a_2 = a_4 + 3 + 2 = a_4 + 5$$
$$< 2^4 + 5 = 16 + 5 < 32 = 2^5.$$

(ii) IF $m = 5$ WE HAVE $a_6 = a_5 + a_4 + a_3$ AND BY

THE INDUCTION HYPOTHESIS, $a_5 < 2^5$, $a_4 < 2^4$.

THEN, $a_6 = a_5 + a_4 + a_3$

$$< 2^5 + 2^4 + 3$$

$$< 2^5 + 2^5 = 2^6.$$

$a_n < 2^n$

$n \in \{1,2,3,4,5\}$

$n \geq 6$

(iii) WE WILL NOW CONSIDER THE CASE $n \geq 6$.

THEN WE OBSERVE

$$4 < n-1 < n, \quad 4 \leq n-2 < n, \quad 4 \leq n = n$$

BY THE INDUCTION HYPOTHESIS, THIS YIELDS

$$a_{n-1} < 2^{n-1}, \quad a_{n-2} < 2^{n-2}, \quad a_n < 2^n$$

HENCE, WE HAVE

$$\begin{aligned} a_{n+1} &= a_n + a_{n-1} + a_{n-2} \\ &< 2^n + 2^{n-1} + 2^{n-2} \\ &< 2^n + 2^{n-1} + 2^{n-1} \\ &= 2^n + 2 \cdot 2^{n-1} = 2^n + 2^n = 2^{n+1}. \end{aligned}$$

THEN, BY THE STRONG PRINCIPLE OF MATHEMATICAL INDUCTION WE HAVE $a_n < 2^n$ FOR ALL $n \geq 1$.

THIS FINISHES THE PROOF. $\blacksquare$

EXERCISE: LET a_n, $n \geq 1$ BE THE SOLUTION TO THE PROBLEM $a_{n+1} = 1 + \frac{n}{a_n}$, $n \geq 1$ WHERE $a_1 = 1$.

SHOW THAT $\sqrt{n} \leq a_n \leq \sqrt{n} + 1$ FOR ALL $n \geq 1$.

PROOF: IF $n = 1$ WE NOTE

$$\sqrt{1} = 1 = a_1 < 2 = \sqrt{1} + 1$$

AND THE RESULT IS TRUE FOR $n = 1$.

ASSUME NEXT THE RESULT IS TRUE FOR SOME $n \in \mathbb{N}$, $n > 1$. WE WANT TO PROVE THAT

$$\sqrt{n+1} \leq a_{n+1} \leq \sqrt{n+1} + 1$$

WE OBSERVE

$$a_{n+1} - \sqrt{n+1} = 1 + \frac{n}{a_n} - \sqrt{n+1}$$

$$= \frac{n}{a_n} - \left(\sqrt{n+1} - 1\right).$$

WE NOW HAVE THE FOLLOWING REMARK:

$$\sqrt{n+1} - 1 = \left(\sqrt{n+1} - 1\right) \cdot \frac{\sqrt{n+1} + 1}{\sqrt{n+1} + 1} = \frac{(n+1) - 1}{\sqrt{n+1} + 1} = \frac{n}{\sqrt{n+1} + 1}.$$

THEN,

$$a_{n+1} - \sqrt{n+1} = \frac{n}{a_n} - \frac{n}{\sqrt{n+1}+1} = \frac{n\left(\sqrt{n+1}+1\right) - a_n \cdot n}{a_n\left(\sqrt{n+1}+1\right)}$$

$$\geq \frac{n\left(\sqrt{n+1}+1\right) - n \cdot \sqrt{n}}{a_n \cdot \left(\sqrt{n+1}+1\right)}$$

BY INDUCTION HYPOTHESIS
$\sqrt{n} \leq a_n$

$$= \frac{\overbrace{n}^{>0}\overbrace{\left(\sqrt{n+1} - \sqrt{n}\right)}^{>0}}{\underbrace{a_n}_{>0}\underbrace{\left(\sqrt{n+1}+1\right)}_{>0}} > 0$$

NOTE $n < n+1 \Rightarrow \sqrt{n} < \sqrt{n+1} \Rightarrow \sqrt{n+1} - \sqrt{n} > 0$

HENCE $a_{n+1} > \sqrt{n+1}$. ALSO,

$$\sqrt{n+1} + 1 - a_{n+1} = \sqrt{n+1} - \frac{n}{a_n}$$

BY HYPOTHESIS
$\sqrt{n} \leq a_n \Rightarrow \frac{1}{a_n} \leq \frac{1}{\sqrt{n}}$

$$\geq \sqrt{n+1} - \frac{n}{\sqrt{n}}$$

$$= \sqrt{n+1} - \sqrt{n} > 0$$

THEN, $a_{n+1} < \sqrt{n+1} + 1$.

SO, WE HAVE SHOWN THAT IF THE RESULT IS TRUE FOR SOME $n \in \mathbb{N}$, $n > 1$, THEN IT IS TRUE FOR $n+1$ AS WELL. THEREFORE, THE RESULT HOLDS FOR ALL $n \geq 1$ BY THE PRINCIPLE OF MATHEMATICAL INDUCTION. $\blacksquare$

EXAMPLE : THE ARITHMETIC - GEOMETRIC MEAN INEQUALITY

LET $a_1, a_2, \dots, a_m$ BE POSITIVE REAL NUMBERS.

THE ARITHMETIC MEAN OF THESE NUMBERS IS DEFINED BY

$$A := \frac{a_1 + a_2 + \dots + a_m}{m}$$

WHILE THE GEOMETRIC MEAN OF THESE NUMBERS IS

$$G := (a_1 \cdot a_2 \cdot \dots \cdot a_m)^{1/m}$$

THE ARITHMETIC - GEOMETRIC MEAN INEQUALITY STATES THAT $G \leq A$ AND THE EQUALITY $G = A$ HOLDS IF AND ONLY IF $a_1 = a_2 = \dots = a_m$.

LET'S PROVE THIS RESULT BY INDUCTION!

WE WILL GIVE A PROOF OF THAT RESULT BY INDUCTION ON m. WE NOTE FIRST THAT IF WE ARE GIVEN ANY POSITIVE REAL NUMBERS $a_1, a_2, \dots, a_m$ WE MAY ASSUME, BY RELABELLING IF NECESSARY, THAT $a_1 \leq a_2 \leq \dots \leq a_m$. IF THIS IS THE CASE THEN CLEARLY $m \cdot a_1 \leq a_1 + a_2 + \dots + a_m \leq m \cdot a_m$.

THIS SHOWS THAT $a_1 \leq \dfrac{a_1 + a_2 + \dots + a_m}{m} \leq a_m$.

WE THUS HAVE $a_1 \leq A \leq a_m$ AND $A - a_1 \geq 0$, $a_m - A \geq 0$.

THEREFORE,

$$(A-a_1)(a_m-A) = A a_m + A a_1 - A^2 - a_1 . a_m \geq 0$$

THIS YIELDS $A\,(a_1 + a_m - A) \geq a_1 . a_m$. $(*)$

BASE CASE: IF $M=1$ THEN $A = \frac{a_1}{1} = (a_1)^1 = G$ AND SO, THE EQUALITY $A = G$ HOLDS.

INDUCTIVE STEP: LET M BE AN ARBITRARY INTEGER $M \geq 2$. SUPPOSE THAT $G \leq A$ IS TRUE FOR ANY SET OF $M-1$ POSITIVE REAL NUMBERS.

LET $a_1, a_2, \ldots, a_m$ BE A SET OF M POSITIVE REAL NUMBERS, LET A DENOTE THEIR ARITHMETIC MEAN AND LET G BE THEIR GEOMETRIC MEAN. WE MAY ASSUME WITHOUT LOSS OF GENERALITY THAT

$$a_1 \leq a_2 \leq \ldots \leq a_{m-1} \leq a_m .$$

WE NEXT CONSIDER THE FOLLOWING $M-1$ POSITIVE REAL NUMBERS: $a_2, a_3, \ldots, a_{m-1}, a_1 + a_m - A$.

SINCE $a_1, a_m, A > 0$, OBSERVE FROM $(*)$ WE HAVE $a_1 + a_m - A > 0$.

THE ARITHMETIC MEAN OF THESE NEW NUMBERS IS

$$\frac{a_2 + a_3 + \dots + a_{m-1} + (a_1 + a_m - A)}{m-1} = \frac{(a_1 + a_2 + \dots + a_m) - A}{m-1}$$

$$= \frac{mA - A}{m-1} = A$$

THIS SHOWS, THEY HAVE THE SAME ARITHMETIC MEAN AS THE ORIGINAL m INTEGERS. BY THE INDUCTIVE HYPOTHESIS,

$$A \geqslant \left(a_2 . a_3 . \dots . a_{m-1} (a_1 + a_m - A)\right)^{\frac{1}{m-1}}$$

SO,

$$A^{m-1} \geqslant a_2 . a_3 . \dots . a_{m-1} (a_1 + a_m - A)$$

IF WE NOW MULTIPLY BY A AND USE $(\star)$,

$$A^m = A^{m-1} . A \geqslant a_2 . a_3 . \dots . a_{m-1} (a_1 + a_m - A) . A$$

$$\geqslant a_2 . a_3 . \dots . a_{m-1} . a_1 . a_m = G^m$$

WE THEREFORE HAVE $G \leq A$.

THIS SHOWS THE INEQUALITY IS ALSO TRUE FOR m WHENEVER IT IS TRUE FOR $m-1$. THIS COMPLETES THE PROOF OF THE INEQUALITY BY INDUCTION.

IT NEEDS TO BE CHECKED THAT $G = A$ HOLDS iff $a_1 = a_2 = \dots = a_m$. THIS REQUIRES SOME EXTRA WORK!

LEMMA: FOR EVERY REAL NUMBER x,

$$e^x \geq 1+x$$

WITH EQUALITY IF AND ONLY IF $x=0$.

THE PROOF OF THE PREVIOUS LEMMA IS NOT DIFFICULT. HOWEVER, SINCE IT REQUIRES SOME TECHNIQUES FROM CALCULUS, WE WILL SKIP IT! YOU CAN DO IT ONCE YOU LEARN ABOUT DERIVATIVES AND THEIR APPLICATIONS. :)

WE WILL USE THE ABOVE LEMMA TO PROVE WHEN THE EQUALITY $G=A$ HOLDS.

LET $a_1, a_2, \dots, a_m$ BE POSITIVE REAL NUMBERS AND LET A BE THEIR ARITHMETIC MEAN. FROM THE PREVIOUS LEMMA THE FOLLOWING EQUALITIES ARE TRUE:

$$e^{\frac{a_1}{A}-1} \geq \frac{a_1}{A}, \quad \dots, \quad e^{\frac{a_m}{A}-1} \geq \frac{a_m}{A}$$

THAT'S IS, TAKING $x=\frac{a_k}{A}-1$, FOR EVERY $1 \leq k \leq m$ WE HAVE

$$e^{\frac{a_k}{A}-1} \geq 1+\left(\frac{a_k}{A}-1\right) = \frac{a_k}{A}$$

MULTIPLYING THESE INEQUALITIES TOGETHER, WE OBTAIN

$$e^{\frac{a_1}{A}-1} \cdot e^{\frac{a_2}{A}-1} \cdot \ldots \cdot e^{\frac{a_m}{A}-1} \geqslant \frac{a_1}{A} \cdot \frac{a_2}{A} \cdot \ldots \cdot \frac{a_m}{A}$$

$$e^{\left(\frac{a_1}{A}-1\right)+\left(\frac{a_2}{A}-1\right)+\ldots+\left(\frac{a_m}{A}-1\right)} \geqslant \frac{a_1 \cdot a_2 \cdot \ldots \cdot a_m}{A^m}$$

WE NEXT OBSERVE:

$$\sum_{k=1}^{m}\left(\frac{a_k}{A}-1\right)=\left(\frac{a_1}{A}-1\right)+\ldots+\left(\frac{a_m}{A}-1\right)=\frac{a_1+\ldots+a_m}{A}-(1+\ldots+1)$$

$$=\frac{m \cdot A}{A}-m \cdot 1=m-m=0.$$

SO, WE HAVE

$$1=e^{0}=e^{\left(\frac{a_1}{A}-1\right)+\ldots+\left(\frac{a_m}{A}-1\right)} \geqslant \frac{a_1 \cdot a_2 \ldots \cdot a_m}{A^m}$$

$$A^m \geqslant a_1 \cdot a_2 \cdot \ldots \cdot a_m = G^m$$

NOTICE THAT TAKING THE m- ROOT WE HAVE $A \geqslant G$ HOLDS WITH EQUALITY IF AND ONLY IF $\frac{a_k}{A}-1=0$ FOR EVERY $1 \leq k \leq m$.

NOTE THE EQUALITY $A = G$ HOLDS IF AND ONLY IF $a_k = A$ FOR EVERY $1 \leq k \leq m$. THIS IS EQUIVALENT TO $a_1 = a_2 = \dots = a_m = A$.