1 Pushdown automata

1.1 Definition

A pushdown automaton is a nondeterministic finite automaton with ε transitions. It also has a *stack* where it can store symbols.

A **pushdown automaton** (PDA) can be defined as $P = (Q, \Sigma, \Gamma, \delta, q_0.Z_0, F)$, where:

- Q is the finite set of states
- Σ : input alphabet (finite set of symbols)
- Γ : stack alphabet (finite set of symbols)
- $q_0 \in Q$: starting state
- $Z_0 \in F$: starting stack symbol
- $F \subseteq Q$: set of accepting states
- $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \rightarrow \mathcal{P}_w(Q \times \Sigma^*)$

1.2 Instantaneous descriptors

Configurations of P: $C = Q \times \Sigma^* \times \Gamma^*$ A configuration can be represented as $(q, w, \gamma) \in C$, where

- q is the current state of P,
- w is the remaining input,
- γ is the string on the stack.

Suppose configuration $(q, aw, Z\gamma)$ and transition $\delta(q, a, Z) = \{..., (p, X), ...\}$. Then

$$(q, aw, Z\gamma) \vdash (p, w, X\gamma)$$

Accepting with final state:

$$L_f = \{w \in \Sigma^* \mid (q_0, w, Z_0) \vdash^* (q, \varepsilon, \delta), \text{ where } q \in F, \delta \in \Sigma^*\}$$

Accepting with empty stack:

$$L_\emptyset = \{w \in \Sigma^* \mid (q_0, w, Z_0) \vdash^* (q, \varepsilon, \varepsilon), \text{ where } q \in Q\}$$

1.3 PDA Exercises

Provide a PDA for the following languages:

1. $L = \{a^n b^{2n} \mid n \geq 0\}$
2. $L = \{wcw^{-1} \mid w \in \{a, b\}^*\}$
3. $L = \{a^{2n} b^n \mid n \geq 0\}$
4. $L = \{a^n b^m | n < m\}$

1.4 CFG to DFA

Converting a CFG grammar G to a PDA is done in the following steps:

1. For each variable A, $\delta(q, \varepsilon, A) = \{(q, \alpha) \mid A \to \alpha \text{ is in } G\}$
2. For each terminal a, $\delta(q, a, a) = \{(q, \varepsilon)\}$

1.5 DFA to grammar

Our grammar will mostly have variables $[pXq]$, that represent changing from state p to q while popping X from the stack. Important that $[pXq]$ is a single variable.

1. For all states p, introduce $S \to [q_0 Z_0 p]$,
2. For each transition $\delta(q, a, X)$ that contains $(r, Y_1 Y_2 ... Y_k)$, introduce $[qXr_k] \to a[rY_1r_1][r_1Y_2r_2]...[r_{k-1}Y_kr_k]$

1.6 Exercise

Consider the following automaton:

$$P = (\{p.q\}, \{0, 1\}, \{X, Z_0\}, \delta, q, Z_0, \emptyset)$$

with transitions:

$$\begin{aligned} \delta(q, 1, Z_0) &\vdash (q, XZ_0), \\ \delta(q, 1, X) &\vdash (q, XX), \\ \delta(q, 0, X) &\vdash (p, X), \\ \delta(q, \varepsilon, X) &\vdash (q, \varepsilon), \\ \delta(p, 1, X) &\vdash (p, \varepsilon), \\ \delta(p, 0, Z_0) &\vdash (q, Z_0) \end{aligned}$$

Transform it to a grammar.