The pumping lemma: If L is a regular language, then there is a number $p > 0$ (sometimes called the pumping length) such that if $w \in L$ and $|w| \geq p$, then w can be written as xyz where

1. $|xy| \leq p$
2. $y \neq \varepsilon$
3. $xy^kz \in L \quad \forall k \geq 0$

Pumping Lemma can be useful to prove that some languages are not regular.

Example 1:
$\Sigma = \{0, 1\}$
$L = \{0^n1^n \mid n \geq 0\}$

Solution:
We will use an indirect proof to show that L is not a regular language. Suppose that L is regular, which means that any string $w \in L$ of length not smaller than p should satisfy the above conditions. Let

$$w = 0^p1^p$$

It can be seen that $|w| = 2p$, which satisfies the condition $|w| \geq p$. If we divide w into parts x, y and z, we know that x and y contain only 0 symbols; $|xy| \leq p$ states that they are at most p symbol long, and our first p symbols are 0s in the string.
Because $y \neq \varepsilon$, it has to contain at least one 0 symbol. Pumping y (xy^kz) for any $k \neq 1$ would result in a new string where the number of 0s is not equal to the number of 1s. (for example, $w_2 = 0^{p-|y|}1^p$ for $k = 0$, $w_3 = 0^{p+|y|}1^p$ for $k = 2$, $w_4 = 0^{p+2|y|}1^p$ for $k = 3$, etc.). As $w_2 \notin L$, we came to a contradiction, meaning that our original assumption is not true; L is not regular.

Example 2:
$\Sigma = \{0, 1\}$
$L = \{ww\}$

Example 3:
$\Sigma = \{1, \#\}$
$L = \{1^i\#1^j\#1^{i+j} \mid i \geq 0\}$

Example 4:
$L = \{w \mid |w| \text{ is prime}\}$