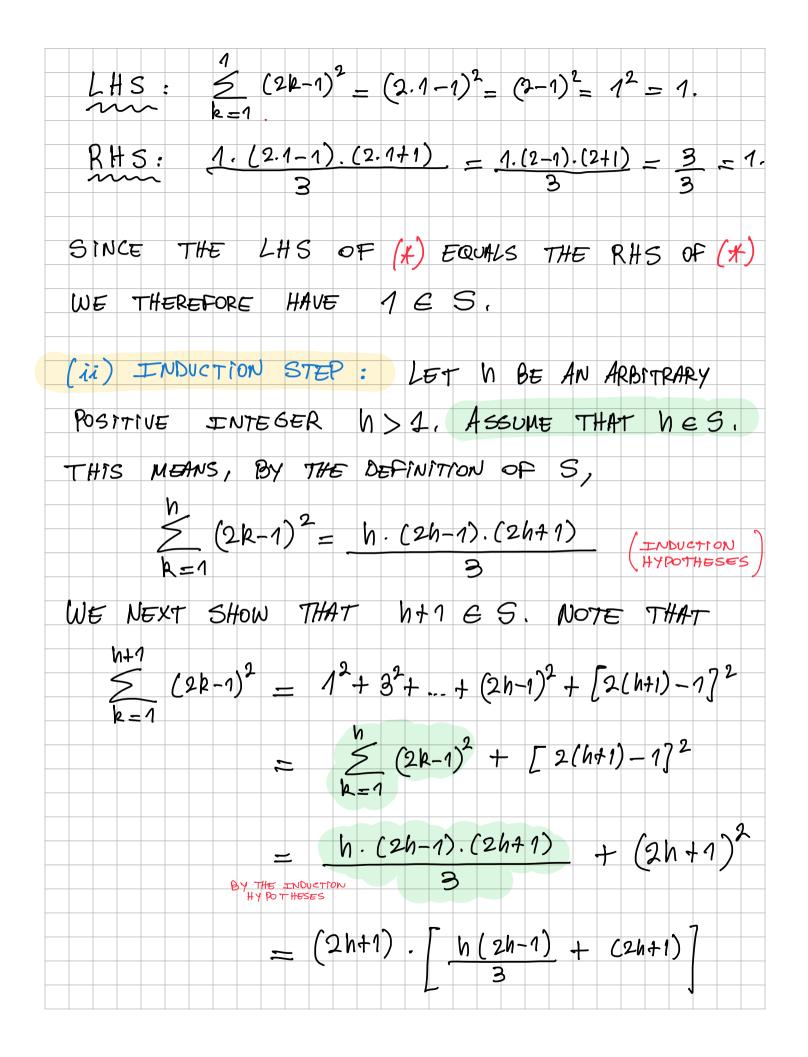
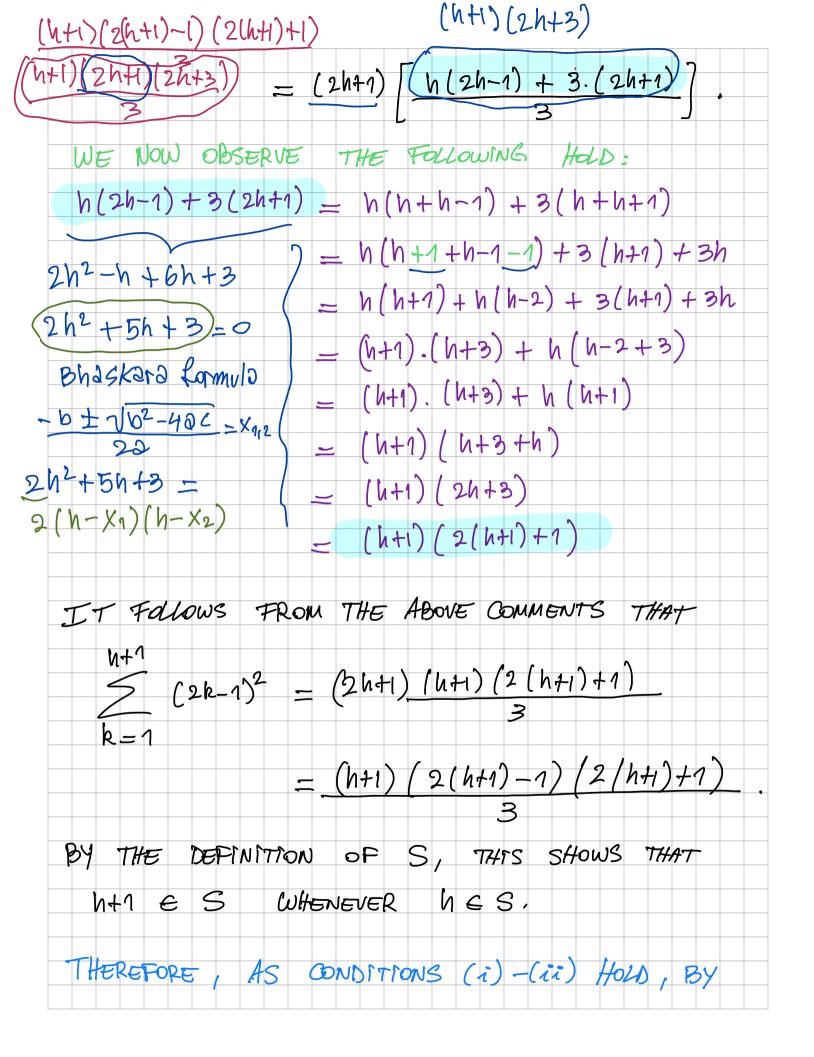
(FRST) MATEMATICAL NOUCTON Principle of Mathematical Induction IF S IS A SET OF INTEGERS SUCH THAT (i)  $a \in S$ (ii) FOR ALL R≥D, IF RES THEN KHIES  $S = \int M \in \mathbb{Z} : M \ge 24$ . THEN WE USUALLY USE MATHEMATICAL INDUCTION FOR ATTEMPTING TO PROVE A STATEMENT ABOUT THE POSITIVE INTEGERS. WHEN GIVING INDUCTION PROOFS, WE OFTEN SHORTEN THE ARGUMENTS BY ELIMINATING ALL REFERENCE TO THE SET S. IT IS ALSO IMPOPTANT TO ESTABLISH BOTH CONDITIONS (i)-(ii) BEFORE GIVING A CONCLUSION. THE PROOF OF CONDITION (i) IS USUALLY CALLED THE BASIS FOR THE INDUCTION, WHILE THE PROOF OF (ii) IS CALLED THE INDUCTION STEP. THE ASSUMPTIONS MADE IN THE INDUCTION STEP ARE KNOWN AS THE INDUCTION HYPOTHESES.

EXERCISE : PROVE BY MATHEMATICAL INDUCTION THAT  $1^{2} + 3^{2} + 5^{2} + \dots + (2M-1)^{2} = M \cdot (2M-1)(2M+1)$ 3 FOR ALL M ≥ 1. PROOF: WE FIRST ABBREVIATE  $1^{2} + 3^{2} + ... + (2M-1)^{2} = \sum_{k=1}^{2} (2k-1)^{2}$ WE NEXT CONSIDER THE SET S DEFINED AS FOLLOWS:  $S = \int_{1}^{1} M \in N$ :  $\sum_{k=1}^{m} (2k-1)^{2} = \frac{M \cdot (2m-1) \cdot (2m+1)}{3}$ BY CONSTRUCTION S & M. WE WOULD LIKE TO PROVE THAT S = IN. TO DO THIS, WE WILL PROCEED BY USING THE FERST PRINCIPLE OF MATHEMATICAL INDUCTION, (i) BASIS FOR THE INDUCTION : WE NEED TO CHECK THAT 1 C S. BY THE DEFINITION OF S, THIS IS EQUIVALENT TO CHECK THE FOLLOWING EQUALITY HOLDS:  $\frac{3}{k} (2k-1)^2 = \frac{1 \cdot (2 \cdot 1 - 1) \cdot (2 \cdot 1 + 1)}{3} \cdot \frac{3}{2}$ (\*)

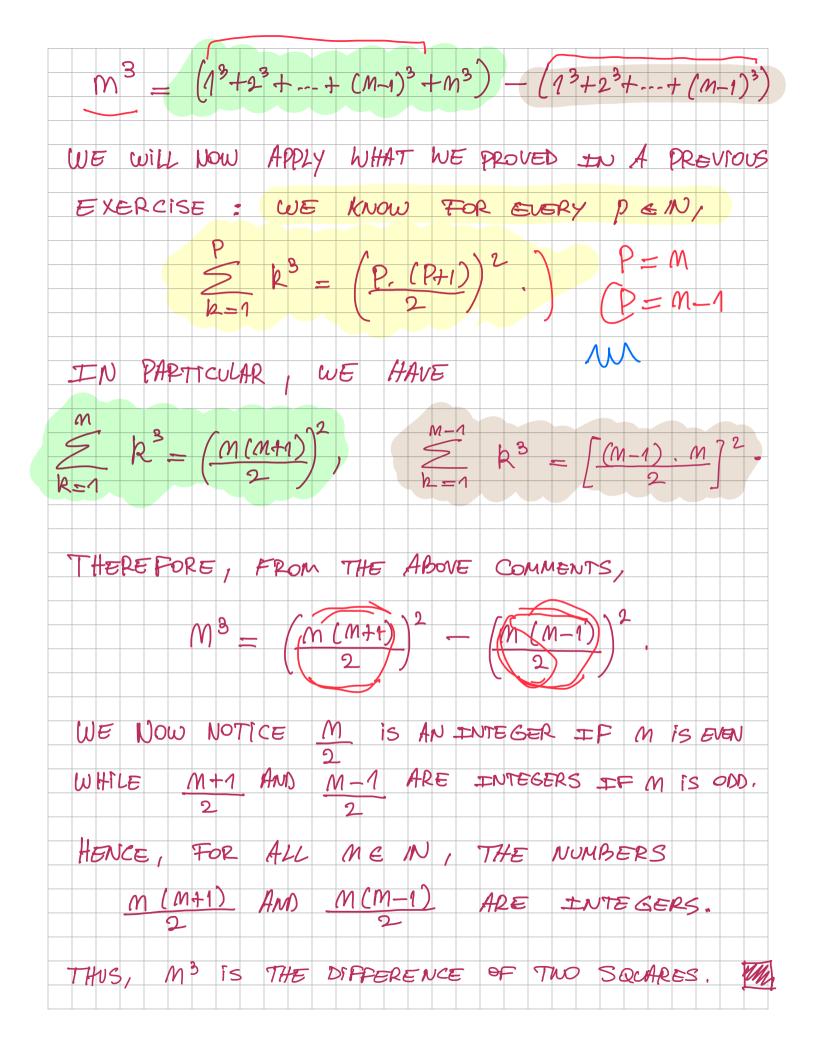


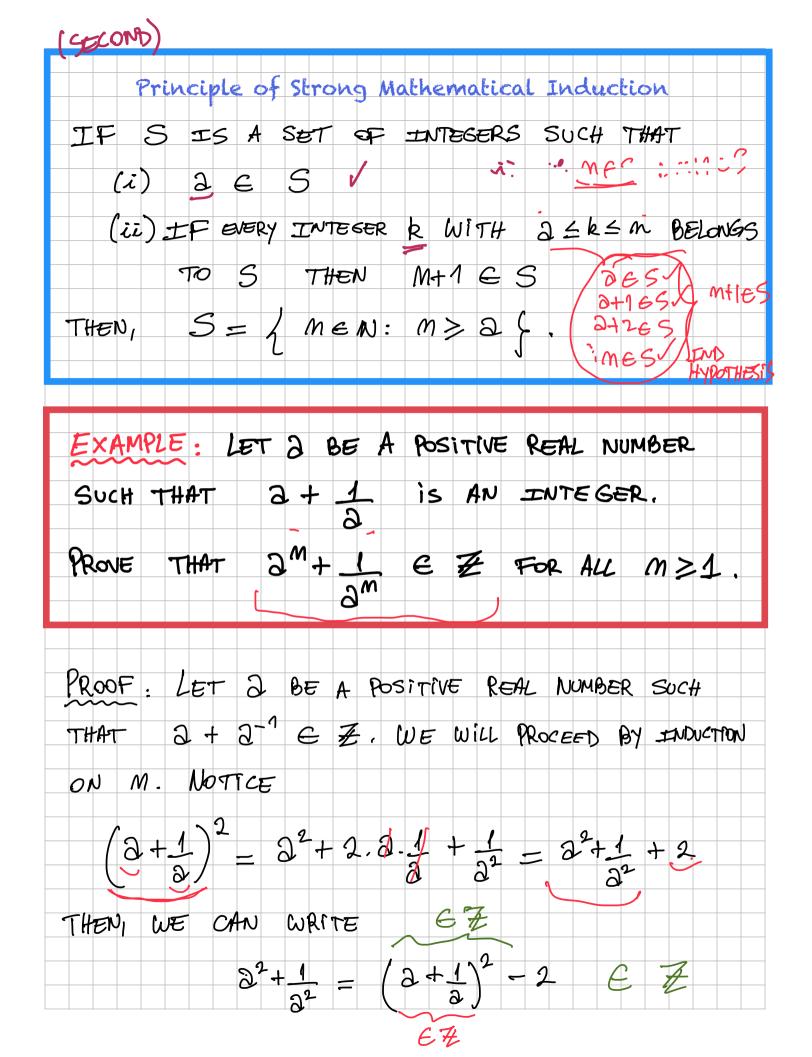


THE FIRST PRINCIPLE OF MATHEMATICAL INDUCTION S=IN. IN THIS CASE, THIS MEANS WE HAVE 3  $(2k-1)^2$  - $M \cdot (2m - 1) \cdot (2m + 1)$  $\forall m \ge 1$ PROOF, THIS OUR CONCLUDES 11h EXERCISE : PROVE BY MATHEMATICAL INDUCTION THAT  $1^{3} + 2^{3} + 3^{3} + \dots + M^{3} = \left( \begin{array}{c} M \cdot (M + 1) \\ 2 \end{array} \right)$ 2 FOR ALL M ≥ 1. PROOF: WE FIRST ABBREVIATE  $1^{3}+2^{3}+3^{3}+\ldots+M^{3}=\sum_{k=1}^{M}k^{3}$ SO, WE NEED TO PROVE FOR ALL  $M \ge 1$ ,  $\sum_{i=1}^{M} k^{3} = \left(\frac{M \cdot (M+1)}{2}\right)^{2}.$  $(\mathbf{X})$ WE WILL PROCEED BY MATHEMATTICAL INDUCTION.

(i) BASIS OF THE INDUCTION: WE NEEDS TO CHECK THAT (+) HOLDS WHEN M=1.  $LHS: \sum_{k=1}^{n} k^{3} = 1^{3} = 1.$  $\frac{\text{RHS}}{2}: \left(\frac{1}{2}, \frac{1+1}{2}\right)^2 = \left(\frac{1}{2}\right)^2 = 1^2 = 1.$ WE THUS HAVE (\*) HOLDS WHEN M=1. (ii) INDUCTION STEP: LET HEN, H>1, ASS ASSUME BY INDUCTION HYPOTHESES THAT (\*) HOLDS WHEN M=h. THAT'S IS,  $\sum_{h=1}^{N} k^3 = \left(\frac{h \cdot (h+1)}{2}\right)^2$ Induction Hypotheses WE NEXT ATTEMPT TO PROVE THE DESIRED EQUALITY FOR N+1 h+1  $k^{3} = 1^{3} + 2^{3} + 3^{3} + \dots + h^{3} + (h+1)^{3}$  $= \sum_{h=1}^{h} k^{3} + (h+1)^{3}$ k = 1 $= \left(\frac{(h.(h+1))^2}{2} + (h+1)^3\right)$ + y potheses

 $\left(h+1\right)^{2}$   $\left[\begin{array}{c}h^{2}\\2^{2}\end{array}$  +  $\left(h+1\right)\right]$  $(h+1)^2$ .  $h^2 + 4(h+1)$ 4  $= (h+1)^2 (h^2 + 4h + 4)$  $= \left[ \frac{(h+1)(h+2)}{2} \right]^{2} = \left[ \frac{(h+1)((h+1)+1)}{2} \right]^{2}$ AS WE WANTED. ACCORDING TO THE PRINCIPLE OF INDUCTION SINCE (i) - (ii) HOLD, WE HAVE THE GIVEN EQUALITY IS VALID FOR ALL MEI, THIS CONCLUDES OUR PROOF. EXERCISE: PROVE THAT THE CUBE OF ANY INTEGER CAN BE WRITTEN AS THE DIFFERENCE  $M^3 = P^2$ TWO SQUARES. of PROOF: WE FIRST NOTICE FOR ALL MEIN,  $m^{3} = 9^{3} + 2^{3} + ... + (m - 1)^{3} + m^{3} - n^{3} - 2^{3} - ... - (m - 1)^{3}$ 



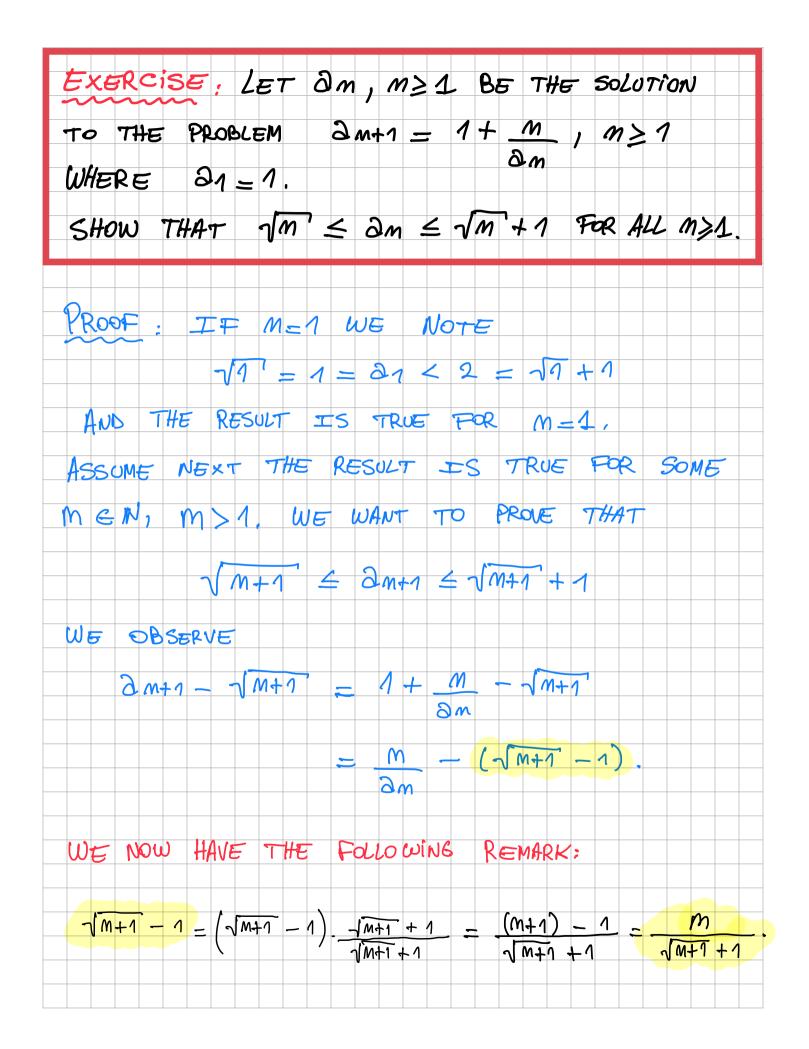


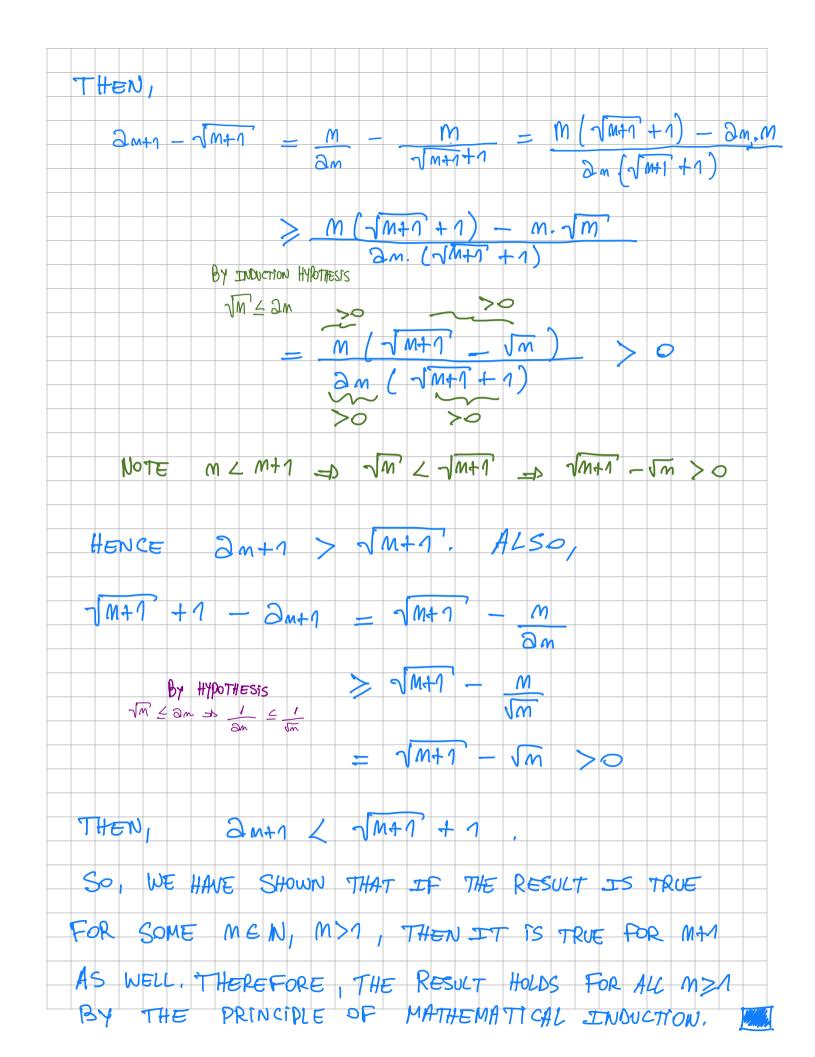
2+1 GZ WE HAVE 22+1 CZ As THE BASE CASE OF THE DUDUETION. THIS PROVES 2+1-EZ FOR ALL ASSUME NEXT THAT INTEGERS 16REM. WE WILL ATTEMPT TO PROVE am+1 WE HAVE Z AS WELL. + E 2<sup>m+1</sup> 2 m +  $a^{m+1} + a^m \cdot 1 + a \cdot 1 + a^m + a^m$ 2+1 -1 2m+1 2 m+1 + 2 m-1 + 6₹ €₹ a<sup>m-1</sup> 2M+1 h=m THEN, WE CAN WRITE 2-1 J m+1  $-\left(\frac{2^{m-1}}{2^{m-1}}\right)$  $\left(\begin{array}{c} a^{M} + f \\ a^{M} \end{array}\right) \cdot \left(\begin{array}{c} a + f \\ a^{M} \end{array}\right) - \left(\begin{array}{c} a + f \\ a^{M} \end{array}\right)$ 1 anti CZ GZ FROM THE INDUCTIVE HYPOTHESIS, THE NUMBERS 2<sup>M</sup>+1, 2+1, 2<sup>M-1</sup>+1 ARE ALL INTEGERS. THAT 2<sup>Md1</sup> + 1 CZ. THIS IMPLIES THEREFORE, BY THE PRINCIPLE OF STRONG MATHEMATICAL INDUCTION, THE NUMBER 2"+1 is An Integer Ъм ALL M ≥ 1. FOR  $25 < 25 = 32^{-1}$ 

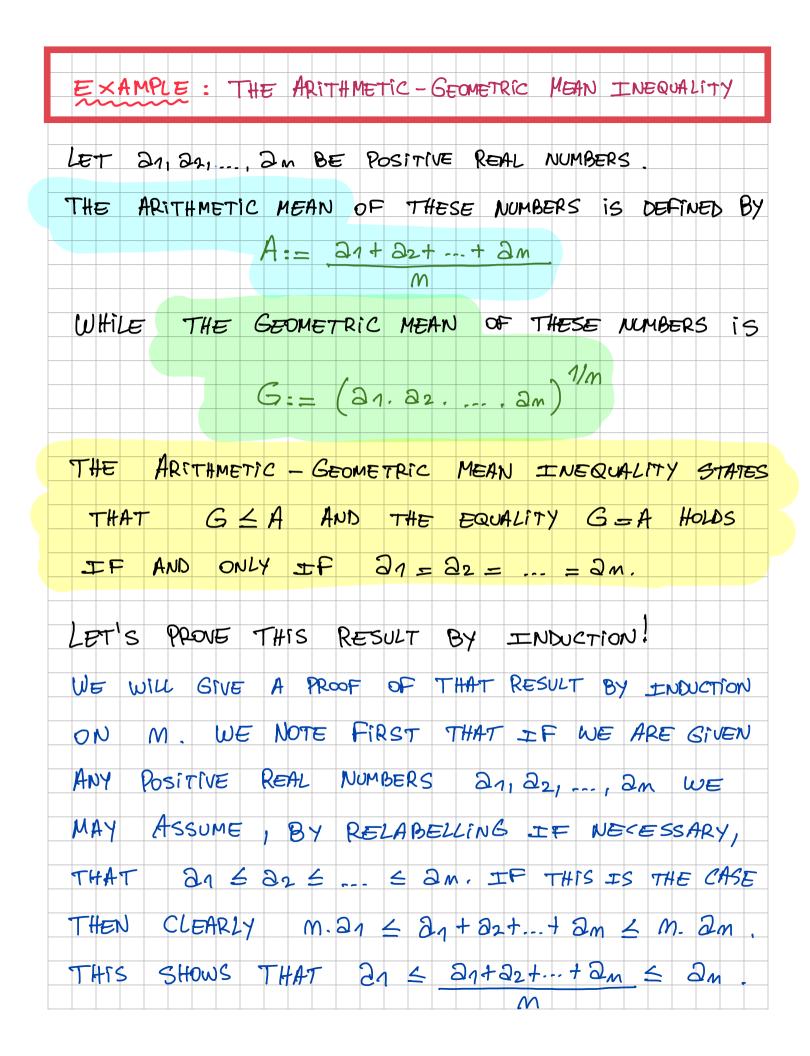
24 = 23 + 22 + 21 = 3 + 241 = 6

25= 24 f2>f22= 6+3+2= 11 246124=46V EXAMPLE : SUPPOSE THAT THE NUMBERS an ARE DEFINED INDUCTIVELY BY 21=1, 22=2, 23-3 AND am = am-1+ am-2 + am-3 For ALL M≥4. PROVE THAT an 22<sup>M</sup> FOR EVERY MEN. PROOF: WE FIRST NOTE an < 2 FOR EVERY MG L11213G. IN FACT, THE INEQUALITIES 142,  $2 \angle 2^2 = 4$ ,  $3 \angle 2^3 = 8$  ARE ALL TRUE.  $2 \angle 2^2 = 4$ ,  $3 \angle 2^3 = 8$  ARE ALL TRUE.  $2 \angle 2^2 = 4$   $3 \angle 2^3 = 8$ LET  $M \ge 4$  AND ASSUME  $\partial_R \angle 2^R$  FOR ALL  $4 \le R \le M$ . WE WILL NEXT SHOW THAT 2MA1 42 MA1. BY DEPINITION WE HAVE  $\partial m + 1 = \partial m + \partial m - 1 + \partial m - 2$ WE NEXT CONSIDER SOME CASES ; (i) IF M=4 WE HAVE  $D_5 = D_4 + D_3 + D_2$ . BY INDUCTION HYPOTHESIS, 24 L 24. RECALL 22=2, 23 = 3. WE THEREFORE HAVE: 25 = 24 + 23 + 22 = 24 + 3 + 2 = 24 + 5  $< 2^{4} + 5 = 16 + 5 \leq 32 = 2^{5}$ . (ii) IF M=5 WE HAVE DG= D5+24+23 AND BY

THE INDUCTION HYPOTHESIS, 25225, 24224. Dm 2 2 ME 1121314,54 MZ6 THEN, 26 = 25 + 24 + 23 $\angle 2^{5} + 2^{4} + 3$  $\angle 2^{5} + 2^{5} = 2^{6}$ (iii) WE WILL NOW CONSIDER THE CASE M26.) THEN WE OBSERVE HLM-1LM, 4LM-2LM, 4LM=M BY THE INDUCTION HYPOTHESIS, THIS YIELDS DM-1 L 2<sup>m-1</sup>, DM-2 L 2<sup>m-2</sup>, DM L 2<sup>m</sup> HENCE, WE HAVE  $\partial m+1 = \partial m + \partial m-1 + \partial m-2$  $< 2^{m} + 2^{m-1} + 2^{m-2}$  $< 2^{m} + 2^{m-1} + 2^{m-1}$   $= 2^{m} + 2 \cdot 2^{m-1} = 2^{m} + 2^{m} = 2^{m+1}$ THEN, BY THE STRONG PRINCIPLE OF MATHEMATICAL INDUCTION WE HAVE IN 2 2 FOR ALL MZ1. THIS FINISHES THE PROOF. 







A-21 20 WE THUS HAVE 21 4 4 2 DM AND 2m-A >0 THERE FORE,  $(A - \partial_1)(\partial_m - A) = A \partial_m + A \partial_1 - A^2 - \partial_1 \partial_m \ge 0$ THIS YIELDS A (21+2m-A) > 21.2m . (\*) BASE CASE: IF M=1 THEN  $A = \underline{a_1} = (\underline{a_1})^2 = G$ AND SO, THE EQUALITY A = G HOLDS. INDUCTIVE STEP : LET M BE AN ARBITRARY INTEGER M>2. SUPPOSE THAT MAN IS TRUE FOR ANY SET OF M-1 POSITIVE REAL NUMBERS. LET 21, 22, ..., 2n BE A SET OF M POSITIVE REAL NUMBERS, LET A DENOTE THEIR ARITHMETIC MEAN AND LET G BE THEIR GEOMETRIC MEAN. WE MAY ASSUME WITHOUT LOSS OF GENERALITY THAT  $a_1 \leq a_2 \leq \ldots \leq a_{m-1} \leq a_m$ WE NEXT CONSIDER THE FOLLOWING M-1 POSITIVE REAL NUMBERS: 22, 23, ..., 2m-1, 21+2m-A. SINCE 21, 2m, A>O, OBSERVE FROM (\*) WE HAVE  $a_1 + a_m - A > 0$ .

THE ARITHMETIC MEAN OF THESE NEW NUMBERS is  $a_{2} + a_{3} + \dots + a_{m-1} + (a_{n} + a_{m} - A) = (a_{n} + a_{2} + \dots + a_{m}) - A$ M-1M-1MA - A =A M-1THIS SHOWS, THEY HAVE THE SAME ARITHMETIC MEAN AS THE ORIGINAL M INTEGERS. BY THE INDUCTIVE HYPOTHESIS,  $A \ge (a_2, a_3, \dots, a_{m-1}(a_n + a_m - A))^{m-1}$ So,  $A \xrightarrow{M-1} \geq \exists_2 \cdot \exists_3 \cdot \ldots \cdot \exists_{M-1} (\exists_1 \neq \exists_m - A)$ IF WE NOW MULTIPLY BY A AND USE (+),  $\geq$   $a_2, a_3, \ldots, a_{m-1}, a_n, a_m = G^m$ WE THEREFORE HAVE G & A. THIS SHOWS THE INEQUALITY IS Also TRUE FOR M WHENEVER IT IS TRUE FOR M-1. THIS COMPLETES THE PROOF OF THE INEQUALITY BY INDUCTION. IT NEEDS TO BE CHECKED THAT G-A HOLDS iff 21=22=... = 2n THIS REQUIRES SOME EXTRA WORK?

