THE BINOMIAL THEOREM

RECALL FOR EVERY POSITIVE INTEGER m, WE DEFINED $m!$ AS FOLLOWS:

$$m! = \begin{cases} 1 & \text{if } m = 1 \\ m.(m-1)! & \text{if } m > 1 \end{cases}$$

THEREFORE WE HAVE $m! = m.(m-1). \ldots .2.1$. BY CONVENTION WE ADDOPT $0! = 1$.

THE FACTORIAL NOTATION HELPS US TO INTRODUCE THE SO CALLED BINOMIAL COEFFICIENTS $\binom{m}{k}$. FOR ANY POSITIVE INTEGER m AND ANY INTEGER k SATISFYING $0 \leq k \leq m$, WE DEFINE

$$\binom{m}{k} := \frac{m!}{k!\,(m-k)!}.$$

THERE ARE MANY IDENTITIES WITH BINOMIAL COEFFICIENTS. ONE OF THEM IS THE WELL-KNOWN PASCAL'S RULE:

$$\binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k} \qquad 1 \leq k \leq m,$$

WE ARE NOW READY TO ANNOUNCE THE BINOMIAL THEOREM WHICH IS IN REALITY A FORMULA FOR THE COMPLETE EXPANSION OF $(a+b)^m$, $m \geq 1$, INTO A SUM OF POWERS OF a AND b:

$$(a+b)^m = \sum_{k=0}^{m} \binom{m}{k} a^k . b^{m-k}.$$

EXERCISE: IF $n \geq 4$ AND $2 \leq k \leq n-2$, SHOW THAT

$$\binom{n}{k} = \binom{n-2}{k-2} + 2 \cdot \binom{n-2}{k-1} + \binom{n-2}{k}.$$

SOLUTION: WE WILL USE THE PASCAL'S RULE. WE OBSERVE

$$\begin{aligned}\binom{n}{k} &= \binom{n-1}{k} + \binom{n-1}{k-1} \\ &= \binom{n-2}{k} + \binom{n-2}{k-1} + \binom{n-2}{k-1} + \binom{n-2}{k-2} \\ &= \binom{n-2}{k} + 2 \cdot \binom{n-2}{k-1} + \binom{n-2}{k-2}.\end{aligned}$$

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EXERCISE: FOR $n \geq 1$, PROVE

$$\binom{n}{0} + 2 \cdot \binom{n}{1} + 2^2 \cdot \binom{n}{2} + \dots + 2^n \cdot \binom{n}{n} = 3^n.$$

SOLUTION: WE WILL USE THE BINOMIAL THEOREM (BT).

WE FIRST OBSERVE

$$\binom{n}{0} + 2 \cdot \binom{n}{1} + 2^2 \cdot \binom{n}{2} + \dots + 2^n \cdot \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k} \cdot 2^k.$$

THEN, NOTICE

$$\sum_{k=0}^{n} \binom{n}{k} \cdot 2^k = \sum_{k=0}^{n} \binom{n}{k} \cdot 2^k \cdot 1^{n-k} \underset{\text{BY BT}}{=} (2+1)^n = 3^n.$$

THE RESULT FOLLOWS.

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EXERCISE: FOR $n \geq 1$, PROVE

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \dots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots = 2^{n-1}.$$

SOLUTION: LET $n \geq 1$. BY THE BINOMIAL THEOREM WE OBSERVE

$$(1) \quad \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} \cdot 1^k \cdot 1^{n-k} = (1+1)^n = 2^n.$$

$$(2) \quad \sum_{k=0}^{n} (-1)^k \cdot \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k \cdot 1^{n-k} = ((-1)+1)^n = 0^n = 0.$$

BY (1) AND (2) ABOVE, WE NEXT OBSERVE

$$2^n = \sum_{k=0}^{n} \binom{n}{k} + \sum_{k=0}^{n} (-1)^k \cdot \binom{n}{k} = \sum_{k=0}^{n} [1+(-1)^k] \cdot \binom{n}{k}.$$

NOTE THAT $1+(-1)^k = \begin{cases} 2 & \text{if } k \text{ is even}, \\ 0 & \text{if } k \text{ is odd}. \end{cases}$

THEREFORE,

$$\sum_{k=0}^{n} [1+(-1)^k] \cdot \binom{n}{k} = \sum_{\substack{0 \leq k \leq n \\ k \text{ even}}} 2 \cdot \binom{n}{k} = 2^n.$$

THIS SHOWS THAT

$$\sum_{\substack{0 \leq k \leq n \\ k \text{ even}}} \binom{n}{k} = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots = \frac{2^n}{2} = 2^{n-1}.$$

SIMILARLY, FROM (1) AND (2) ABOVE WE HAVE

$$2^n = \sum_{k=0}^{n} \binom{n}{k} - \sum_{k=0}^{n} (-1)^k \cdot \binom{n}{k} = \sum_{k=0}^{n} [1-(-1)^k] \cdot \binom{n}{k}$$

OBSERVE THAT $1-(-1)^k = \begin{cases} 0 & \text{if } k \text{ is even}, \\ 2 & \text{if } k \text{ is odd}. \end{cases}$

THEN,

$$\sum_{k=0}^{n} [1-(-1)^k] \cdot \binom{n}{k} = \sum_{\substack{0 \leq k \leq n \\ k \text{ odd}}} 2 \cdot \binom{n}{k} = 2^n.$$

WE THUS HAVE

$$\sum_{\substack{0\le k\le m \\ k \text{ odd}}} \binom{m}{k} = \binom{m}{1}+\binom{m}{3}+\binom{m}{5}+\dots = \frac{2^m}{2} = 2^{m-1}.$$

THE RESULT FOLLOWS. ■

EXERCISE: DOES EXIST $m \in \mathbb{N}$ SUCH THAT

$$1-\frac{1}{2}m+\frac{1}{6}m\cdot(m-1)-\dots+\frac{(-1)^m}{m+1} = \frac{1}{2021}\ ?$$

SOLUTION: WE FIRST TRY TO REWRITE THE LHS OF THE ABOVE EQUALITY USING BINOMIAL COEFFICIENTS. NOTE THAT $\binom{m}{0}=1$, $\binom{m}{1}=m$, $\binom{m}{2}=\frac{m(m-1)}{2}$, $\binom{m}{m}=1$.

THEN, WE CAN WRITE

$$1-\frac{1}{2}m+\frac{1}{6}m(m-1)-\dots+\frac{(-1)^m}{m+1} = \binom{m}{0}-\frac{1}{2}\binom{m}{1}+\frac{1}{3}\binom{m}{2}-\dots+\frac{(-1)^m}{m+1}\binom{m}{m}$$

SO, OUR PROBLEM IS TO FIND $m \in \mathbb{N}$ (IF THERE EXISTS) SUCH THAT

$$\sum_{k=0}^{m} \frac{(-1)^k}{k+1}\cdot\binom{m}{k} = \frac{1}{2021}.$$

TO DO THAT, WE OBSERVE

$$\frac{1}{k+1}\cdot\binom{m}{k} = \frac{1}{m+1}\cdot\frac{(m+1)}{k+1}\cdot\binom{m}{k}$$

$$= \frac{1}{m+1}\cdot\frac{(m+1)}{k+1}\cdot\frac{m!}{k!\,(m-k)!}$$

$$= \frac{1}{m+1}\cdot\frac{(m+1)!}{(k+1)!\,(m-k)!} = \frac{1}{m+1}\cdot\binom{m+1}{k+1}.$$

WE THEREFORE HAVE

$$\sum_{k=0}^{m} \frac{(-1)^k}{k+1}\binom{m}{k} = \sum_{k=0}^{m} \frac{(-1)^k}{m+1}\binom{m+1}{k+1} =$$

$$(j=k+1) = \sum_{j=1}^{m+1} \frac{(-1)^{j-1}}{m+1}\cdot\binom{m+1}{j}$$

$$= -\sum_{j=1}^{m+1} \frac{(-1)^j}{m+1}\cdot\binom{m+1}{j}.$$

$$= -\frac{1}{m+1}\sum_{j=1}^{m+1}(-1)^j\cdot\binom{m+1}{j}$$

WE NOW OBSERVE

$$\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j} = (-1)^0\cdot\binom{m+1}{0} + \sum_{j=1}^{m+1}(-1)^j\binom{m+1}{j}$$

$$= 1 + \sum_{j=1}^{m+1}(-1)^j\cdot\binom{m+1}{j}$$

THEN,

$$\sum_{j=1}^{m+1}(-1)^j\binom{m+1}{j} = \sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j} - 1$$

IT FOLLOWS FROM THE BINOMIAL THEOREM THAT

$$\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j} = \sum_{j=0}^{m+1}\binom{m+1}{j}(-1)^j\cdot 1^{m+1-j} = ((-1)+1)^m = 0^m = 0.$$

HENCE WE GET $\displaystyle\sum_{j=1}^{m+1}(-1)^j\binom{m+1}{j} = -1.$

THIS IMPLIES

$$\sum_{k=0}^{m} \frac{(-1)^k}{k+1} \cdot \binom{m}{k} = \frac{-1}{m+1} \cdot \sum_{j=1}^{m+1} (-1)^j \binom{m+1}{j} = \frac{1}{m+1}.$$

THEN, $m = 2020$ SATISFIES WHAT WE WANTED:

$$\sum_{k=0}^{2020} \frac{(-1)^k}{k+1} \binom{m}{k} = \frac{1}{2020+1} = \frac{1}{2021}. \blacksquare$$

EXERCISE: FIND THE COEFFICIENT OF x^{11} IN THE EXPANSION OF $\left(x^3 - \frac{2}{x^2}\right)^{12}$.

SOLUTION: BY THE BINOMIAL THEOREM,

$$\left(x^3 - \frac{2}{x^2}\right)^{12} = \sum_{k=0}^{12} \binom{12}{k} \cdot (x^3)^k \cdot \left(\frac{-2}{x^2}\right)^{12-k}$$

$$= \sum_{k=0}^{12} \binom{12}{k} \cdot x^{3k} \cdot (-2)^{12-k} \cdot (x^{-2})^{12-k}$$

$$= \sum_{k=0}^{12} \binom{12}{k} \cdot (-2)^{12-k} \cdot x^{3k+(-2)\cdot(12-k)}$$

THEN, WE NEED TO FIND $k \in \mathbb{N}$ SUCH THAT $3k+(-2)(12-k)=11$.

NOTE $3k+(-2)(12-k) = 3k - 24 + 2k = 5k - 24$. SO,

$5k - 24 = 11 \iff 5k = 35 \iff k = 7.$

THEREFORE, THE COEFFICIENT OF x^{11} EQUALS

$$\binom{12}{7}(-2)^{12-7} = \binom{12}{7} \cdot (-2)^5 = 792 \cdot (-32) = -25344. \blacksquare$$

EXERCISE : FIND THE COEFFICIENT OF $a^5 b^4$ IN THE EXPANSION OF $\left(3a - \frac{b}{3}\right)^9$.

SOLUTION: BY THE BINOMIAL THEOREM,

$$\left(3a - \frac{b}{3}\right)^9 = \sum_{k=0}^{9} \binom{9}{k} (3a)^k \left(-\frac{b}{3}\right)^{9-k}$$

$$= \sum_{k=0}^{9} \binom{9}{k} 3^k \cdot a^k \cdot \frac{(-1)^{9-k} \cdot b^{9-k}}{3^{9-k}}$$

$$= \sum_{k=0}^{9} \binom{9}{k} \cdot 3^{2k-9} \cdot (-1)^{9-k} \cdot a^k b^{9-k}$$

NOTE THE COEFFICIENT OF $a^5 b^4$ IS OBTAINED IN THE ABOVE SUM WHEN $k=5$. WE THEREFORE HAVE SUCH COEFFICIENT IS

$$\binom{9}{5} \cdot 3^{2.5-9} \cdot (-1)^{9-5} = \binom{9}{5} \cdot 3 \cdot (-1)^4 = 126 \cdot 3 = 378.$$

$\blacksquare$

MIDDLE TERMS: THE MIDDLE TERM DEPENDS UPON THE VALUE OF m. IF m IS EVEN, THEN THE TOTAL NUMBER OF TERMS IN THE EXPANSION OF $(a+b)^m$ IS $m+1$ (odd). HENCE, THERE IS ONLY ONE MIDDLE TERM: THE $\left(\frac{m}{2}+1\right)$-TH TERM.

IF m IS ODD, THEN THE TOTAL NUMBER OF TERMS IN THE EXPANSION OF $(a+b)^m$ IS $m+1$ (even). SO, THERE ARE TWO MIDDLE TERMS: THE $\left(\frac{m+1}{2}\right)$-TH TERM AND THE $\left(\frac{m+3}{2}\right)$-TH TERM.

EXERCISE: FIND THE MIDDLE TERM IN THE EXPANSION OF $\left(2a x - \frac{b}{x^2}\right)^{12}$.

SOLUTION: SINCE THE POWER OF THE BINOMIAL IS EVEN, IT HAS JUST ONE MIDDLE TERM WHICH IS THE 7^{TH} TERM. NOTE $7 = \frac{12}{2} + 1$. BY THE BINOMIAL THEOREM,

$$\left(2ax - \frac{b}{x^2}\right)^{12} = \sum_{k=0}^{12} \binom{12}{k} (2ax)^k \left(\frac{-b}{x^2}\right)^{12-k}$$

$$= \sum_{k=0}^{12} \binom{12}{k} 2^k . a^k . x^k . (-1)^{12-k} . b^{12-k} . \left(x^{-2}\right)^{12-k}$$

$$= \sum_{k=0}^{12} \binom{12}{k} . 2^k . (-1)^{12-k} . a^k . b^{12-k} . x^{3k-24}$$

THEREFORE, THE MIDDLE TERM IS OBTAINED IN THE ABOVE EXPRESSION WHEN $k = 7-1 = 6$. SO, WE GET

$$\binom{12}{6} . 2^6 . a^6 . b^6 . x^{-6} = \frac{59136\, a^6 b^6}{x^6}. \qquad \blacksquare$$

EXERCISE: FIND THE MIDDLE TERM IN THE EXPANSION OF $\left(\frac{p}{x} + \frac{x}{p}\right)^9$.

SOLUTION: SINCE THE POWER OF THE BINOMIAL IS ODD, WE HAVE TWO MIDDLE TERMS WHICH ARE THE 5^{TH} AND 6^{TH} TERMS. BY THE BINOMIAL THEOREM, WE OBSERVE

$$\left(\frac{p}{x} + \frac{x}{p}\right)^9 = \sum_{k=0}^{9} \binom{9}{k} \left(\frac{p}{x}\right)^k . \left(\frac{x}{p}\right)^{9-k}$$

$$= \sum_{k=0}^{9} \binom{9}{k} \frac{p^k}{p^{9-k}} \cdot \frac{x^{9-k}}{x^k}$$

$$= \sum_{k=0}^{9} \binom{9}{k} \cdot p^{2k-9} \cdot x^{9-2k}.$$

THE 5^{TH}-TERM IS OBTAINED IN THE ABOVE EXPRESION WHEN $k=4$. THAT IS,

$$\binom{9}{4} \frac{x}{p} = 126 \frac{x}{p}.$$

SIMILARLY, THE 6^{TH}-TERM IS OBTAINED WHEN $k=5$ AND IS GIVEN BY

$$\binom{9}{5} \frac{p}{x} = 126 \frac{p}{x}. \qquad \blacksquare$$

EXERCISE: FIND THE NUMBER OF TERMS IN THE EXPANSION OF $(x+y+z)^{2021}$.

SOLUTION: WE WILL COMPUTE THE NUMBER OF TERMS IN THE EXPANSION OF $(x+y+z)^m$ FOR AN ARBITRARY $m \in \mathbb{N}$. BY THE BINOMIAL THEOREM WE OBSERVE

$$(x+y+z)^m = (x + (y+z))^m = \sum_{k=0}^{m} \binom{m}{k} x^k (y+z)^{m-k}$$

$$= \sum_{k=0}^{m} \binom{m}{k} x^k \cdot \left(\sum_{j=0}^{m-k} \binom{m-k}{j} y^j \cdot z^{m-k-j} \right)$$

$$= \sum_{k=0}^{m} \sum_{j=0}^{m-k} \binom{m}{k} \binom{m-k}{j} x^k \cdot y^j \cdot z^{m-k-j}$$

SUPPOSE NOW THE VALUE OF k IS FIXED. THEN, WE HAVE $m-k+1$ POSSIBLE CHOICES FOR j. ONCE WE CHOOSE

THE VALUES OF k AND j, THE VALUE OF THE POWER OF z IS DETERMINED SINCE WE KNOW m, k, j.

IT FOLLOWS FROM THE ABOVE COMMENTS THAT THE NUMBER OF TERMS IS GIVEN AS FOLLOWS:

$$\sum_{k=0}^{m} (m-k+1) = \sum_{k=0}^{m} (m+1) - \sum_{k=0}^{m} k$$

$$= (m+1)(m+1) - \frac{m(m+1)}{2}$$

$$= (m+1) \cdot \left[(m+1) - \frac{m}{2} \right]$$

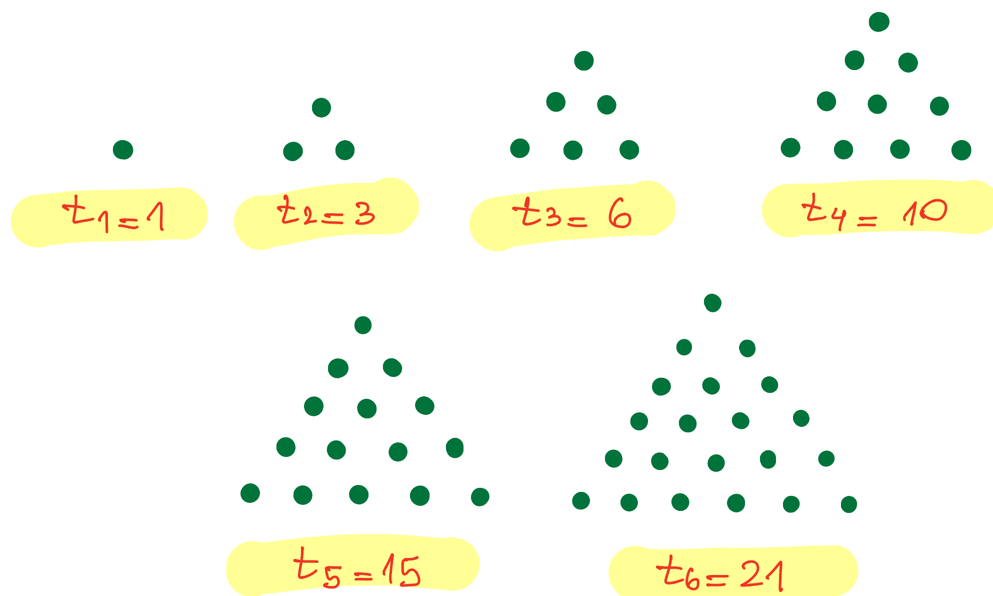
$$= \frac{(m+1)(m+2)}{2}.$$

THEREFORE, WHEN $m = 2021$, THE NUMBER OF TERMS IN THE EXPANSION OF $(x+y+z)^{2021}$ IS

$$\frac{(2021+1)(2021+2)}{2} = \frac{2022 \cdot 2023}{2} = 2045253. \qquad \blacksquare$$

Exercises: Triangular numbers

(1) EACH OF THE NUMBERS 1, 3, 6, 10, 15, 21 ... REPRESENTS THE NUMBER OF DOTS THAT CAN BE ARRANGED EVENLY IN AN EQUILATERAL TRIANGLE:



THIS LED THE ANCIENT GREEKS TO CALL A NUMBER <u>TRIANGULAR</u> IF IT IS THE SUM OF CONSECUTIVE INTEGERS, BEGINNING WITH 1. WE OBSERVE

$t_1 = 1$

$t_2 = 1 + 2 = 3$

$t_3 = 1 + 2 + 3 = 6$

$t_4 = 1 + 2 + 3 + 4 = 10$

$t_5 = 1 + 2 + 3 + 4 + 5 = 15$

$t_6 = 1 + 2 + 3 + 4 + 5 + 6 = 21$

PROVE THE FOLLOWING FACTS CONCERNING TRIANGULAR NUMBERS:

(a) A NUMBER IS TRIANGULAR IF AND ONLY IF IT IS OF

THE FORM $\frac{m(m+1)}{2}$ FOR SOME $m \geq 1$

SOLUTION:

($\Rightarrow$) LET t BE A TRIANGULAR NUMBER. THEN, BY DEFINITION, THERE EXISTS $m \in \mathbb{N}$ SUCH THAT $t = 1+2+\dots+(m-1)+m$ WHICH MEANS THAT $t = \sum_{k=1}^{m} k = \frac{m(m+1)}{2}$.

($\Leftarrow$) CONVERSELY, IF FOR SOME $m \geq 1$, WE HAVE $t = \frac{m\cdot(m+1)}{2}$ THEN $t = 1+2+\dots+m$ WHICH MEANS THAT t IS SUM OF CONSECUTIVE INTEGERS, BEGINNING WITH 1. THEN, t IS TRIANGULAR.

(b) THE INTEGER m IS TRIANGULAR IF AND ONLY IF $8m+1$ IS A PERFECT SQUARE.

($\Rightarrow$) SUPPOSE FIRST THAT m IS TRIANGULAR. THEN, THERE EXISTS $t \in \mathbb{N}$ SUCH THAT $m = \frac{t.(t+1)}{2}$. THIS SHOWS THAT $8m+1 = 4t(t+1)+1 = 4t^2+4t+1 = (2t+1)^2$. SINCE $2t+1 \in \mathbb{N}$, IT FOLLOWS THAT $8m+1$ IS A PERFECT SQUARE.

($\Leftarrow$) ASSUME NOW THAT $8m+1$ IS A PERFECT SQUARE. THEN, THERE EXISTS AN INTEGER p SUCH THAT $p^2 = 8m+1$. NOTE THAT $8m+1$ IS odd. THEN p^2 IS odd AND SO, THERE EXISTS $q \in \mathbb{N}$ SUCH THAT $p = 2q+1$. HENCE, $(2q+1)^2 = 8m+1$ IMPLIES $4q^2+4q+1 = 8m+1$ WHICH YIELDS $4q^2+4q = 8m$.

WE THEREFORE HAVE

$$m = \frac{49q^2+49q}{8} = \frac{49q(q+1)}{8} = \frac{q(q+1)}{2}.$$

THIS MEANS THAT m IS TRIANGULAR.

(c) THE SUM OF ANY TWO CONSECUTIVE TRIANGULAR NUMBERS IS A PERFECT SQUARE.

SOLUTION: LET p AND q BE TWO CONSECUTIVE TRIANGULAR NUMBERS. THEN, THERE EXISTS $m \in \mathbb{N}$ SUCH THAT $p = \frac{m(m+1)}{2}$ AND $q = p + (m+1) = \frac{m(m+1)}{2} + (m+1) = \frac{(m+1)(m+2)}{2}$.

THEN, WE OBSERVE

$$p+q = \frac{m(m+1)}{2} + \frac{(m+1)(m+2)}{2} = \frac{(m+1)}{2}\left[m + (m+2)\right]$$
$$= \frac{(m+1)}{2} \cdot 2(m+1) = (m+1)^2.$$

THEREFORE, $p+q$ IS A PERFECT SQUARE.

(d) IF m IS TRIANGULAR THEN $25m+3$ IS TRIANGULAR AS WELL.

SOLUTION: SUPPOSE THAT m IS TRIANGULAR. THEN, THERE EXISTS $t \in \mathbb{N}$ SUCH THAT $m = \frac{t\cdot(t+1)}{2}$. SO,

$$25m+3 = \frac{25t(t+1)}{2} + 3 = \frac{25t(t+1)+6}{2} = \frac{25t^2+25t+6}{2}.$$

NOTICE $25t^2+25t+6 = 5t(5t+2) + 15t+6$

$$= 5t(5t+2) + 3(5t+2)$$
$$= (5t+2)(5t+3) = n(n+1)$$

WHERE $n = 5t+2 \in \mathbb{N}$.

THEREFORE, $25n+3 = \frac{m.(m+1)}{2}$ AND SO, $25n+3$ IS TRIANGULAR.

(e) IF t_n DENOTES THE n-TH TRIANGULAR NUMBER, PROVE $t_n = \binom{n+1}{2}$ FOR $n \geq 1$.

SOLUTION: BY DEFINITION,

$$t_n = 1+2+\dots+n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \binom{n+1}{2}.$$

(f) FIND THE SUM OF THE FIRST 2021 TRIANGULAR NUMBERS.

SOLUTION: LET t_m DENOTE THE m-TH TRIANGULAR NUMBER.

LET S_{2021} DENOTE THE SUM OF THE FIRST 2021 TRIANGULAR NUMBERS. WE NEED TO FIND

$$S_{2021} = t_1 + t_2 + t_3 + \dots + t_{2020} + t_{2021} = \sum_{k=1}^{2021} t_k$$

BY THE PREVIOUS EXERCISE WE HAVE

$$S_{2021} = \sum_{k=1}^{2021} t_k = \sum_{k=1}^{2021} \binom{k+1}{2} = \sum_{k=1}^{2021} \frac{(k+1)k}{2} = \frac{1}{2} \cdot \sum_{k=1}^{2021} (k^2+k)$$

WE THEREFORE HAVE

$$S_{2021} = \frac{1}{2} \cdot \sum_{k=1}^{2021} k^2 + \frac{1}{2} \cdot \sum_{k=1}^{2021} k.$$

RECALL FOR EVERY $m \in \mathbb{N}$ WE HAVE $\sum_{k=1}^{m} k = \frac{m.(m+1)}{2}$.

SO, WE JUST NEED TO FIND A FORMULA FOR

$$\sum_{k=1}^{m} k^2.$$

TO DO THAT WE OBSERVE

$$(k-1)^3 = k^3 - 3k^2 + 3k - 1 \Rightarrow \boxed{k^3 - (k-1)^3 = 3k^2 - 3k + 1}$$

THEN,
$$\sum_{k=1}^{m} \left(k^3 - (k-1)^3\right) = \sum_{k=1}^{m} \left(3k^2 - 3k + 1\right).$$

NOTE THAT
$$\sum_{k=1}^{m} k^3 - (k-1)^3 = \sum_{k=1}^{m} k^3 - \sum_{k=1}^{m} (k-1)^3$$
$$= \sum_{k=1}^{m} k^3 - \sum_{j=0}^{m-1} j^3 \qquad (j = k-1)$$
$$= \cancel{\sum_{k=1}^{m-1} k^3} + m^3 - 0^3 - \cancel{\sum_{j=1}^{m-1} j^3} = m^3.$$

SO, WE HAVE
$$m^3 = \sum_{k=1}^{m} \left[k^3 - (k-1)^3\right] = \sum_{k=1}^{m} \left(3k^2 - 3k + 1\right)$$
$$= 3\sum_{k=1}^{m} k^2 - 3\sum_{k=1}^{m} k + \sum_{k=1}^{m} 1$$
$$= 3 \cdot \sum_{k=1}^{m} k^2 - 3 \cdot \frac{m \cdot (m+1)}{2} + m \cdot 1$$

THEREFORE,
$$3 \cdot \sum_{k=1}^{m} k^2 = m^3 + 3\,\frac{m(m+1)}{2} - m$$
$$= m \cdot \left[m^2 + \frac{3(m+1)}{2} - 1\right]$$
$$= m \cdot \left[\frac{2(m^2-1) + 3(m+1)}{2}\right]$$
$$= m\left[\frac{2(m+1)(m-1) + 3(m+1)}{2}\right]$$

$$= \frac{n(n+1)\cdot[2(n-1)+3]}{2}$$

$$= \frac{n(n+1)(2n+1)}{2}$$

THIS SHOWS $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

HENCE, WE GET

$$S_{2021} = \frac{1}{2}\cdot\sum_{k=1}^{2021} k^2 + \frac{1}{2}\cdot\sum_{k=1}^{2021} k.$$

$$= \frac{1}{2}\cdot\frac{2021\cdot 2022\cdot(2\cdot 2021+1)}{6} + \frac{1}{2}\cdot\frac{2021\cdot 2022}{2}$$

$$= \frac{1}{2}\cdot\frac{2021\cdot 2022}{2}\cdot\left(\frac{2\cdot 2021+1}{3} + 1\right)$$

$$= \frac{1}{2}\cdot\frac{2021\cdot 2022}{2}\cdot\frac{2\cdot 2021+1+3}{3}$$

$$= \frac{1}{2}\cdot\frac{2021\cdot 2022}{\cancel{2}}\cdot\frac{\cancel{2}(2021+2)}{3}$$

$$= \frac{2021\cdot 2022\cdot 2023}{6} \qquad \blacksquare$$

(g) PROVE THAT THE SUM OF THE RECIPROCALS OF THE FIRST n TRIANGULAR NUMBERS IS LESS THAN 2.

SOLUTION: LET t_n DENOTE THE n-TH TRIANGULAR NUMBER. OBSERVE ITS RECIPROCAL IS $1/t_n$. RECALL ALSO THAT $t_n = \binom{n+1}{2} = \frac{n(n+1)}{2}$.

THEREFORE, WE HAVE $\frac{1}{t_m} = \frac{2}{m(m+1)}$.

WE NEXT CLAIM THAT $\frac{1}{t_m} = \frac{2}{m} - \frac{2}{m+1}$.

TO PROVE OUR CLAIM, NOTE THAT

$$\frac{1}{m(m+1)} = \frac{A}{m} + \frac{B}{m+1} = \frac{A(m+1)+Bm}{m(m+1)}$$

$$= \frac{(A+B)m + A}{m(m+1)}$$

THIS MEANS THAT $\begin{cases} A+B=0 \\ A=1 \end{cases}$ WHICH YIELDS $A=1$, $B=-1$. THEN, WE SEE THAT

$$\boxed{\frac{1}{m(m+1)} = \frac{1}{m} - \frac{1}{m+1}}$$

SO, THIS IMPLIES THAT

$$\boxed{\frac{1}{t_m} = \frac{2}{m(m+1)} = \frac{2}{m} - \frac{2}{m+1} \quad \forall m \in \mathbb{N}}$$

COMING BACK TO OUR MAIN QUESTION, WE NEED TO FIND

$$\sum_{k=1}^{m} \frac{1}{t_k} = \sum_{k=1}^{m} \frac{2}{k(k+1)} = \sum_{k=1}^{m} \left(\frac{2}{k} - \frac{2}{k+1} \right)$$

$$= \sum_{k=1}^{m} \frac{2}{k} - \sum_{k=1}^{m} \frac{2}{k+1}$$

$$= \frac{2}{1} + \frac{2}{2} + \frac{2}{3} + \dots + \frac{2}{m-1} + \frac{2}{m}$$

$$- \frac{2}{2} - \frac{2}{3} - \dots - \frac{2}{m-1} - \frac{2}{m} - \frac{2}{m+1}$$

$$= 2 - \frac{2}{m+1} = 2\left(1 - \frac{1}{m+1}\right)$$

$$= 2\left(\frac{m+1-1}{m+1}\right) = \frac{2m}{m+1}.$$

NOTE THAT $m < m+1$ AND SO $\frac{m}{m+1} < 1$.

WE THUS HAVE

$$\sum_{k=1}^{m} \frac{1}{t_k} = \frac{2m}{m+1} = 2 \cdot \frac{m}{m+1} < 2 \cdot 1 = 2.$$

THIS COMPLETES THE PROOF. ■

(h) LET t_m DENOTE THE m-TH TRIANGULAR NUMBER.

FIND $t_{1000} + t_{1002} + t_{1004} + \dots + t_{2020} + t_{2021}$.

SOLUTION: RECALL THAT $t_m = \binom{m+1}{2}$ FOR EVERY $m \in \mathbb{N}$.

SO, $t_{2m} = \binom{2m+1}{2} = \frac{(2m+1) \cdot 2m}{2} = (2m+1) \cdot 2m$.

WE THEREFORE HAVE $t_{2m} = 4m^2 + 2m$.

NOW WE OBSERVE

$$t_{1000} + t_{1002} + t_{1004} + \dots + t_{2020} + t_{2021} = \sum_{k=500}^{1010} t_{2k} + t_{2021}$$

WE FIRST COMPUTE THE VALUE OF $\sum_{k=500}^{1010} t_{2k}$.

NOTICE THAT

$$\sum_{k=500}^{1010} t_{2k} = \sum_{k=500}^{1010} (4k^2+2k) = 4 \cdot \sum_{k=500}^{1010} k^2 + 2 \cdot \sum_{k=500}^{1010} k .$$

RECALL FROM PREVIOUS EXERCISES THAT WE ALREADY KNOW FORMULAS FOR THE SUM OF THE FIRST m SQUARES NUMBERS AND FOR THE SUM OF THE FIRST m NATURAL NUMBERS. NAMELY, FOR EVERY $m \in \mathbb{N}$, WE HAVE

$$\boxed{\sum_{k=1}^{m} k = \frac{m(m+1)}{2}} \qquad \boxed{\sum_{k=1}^{m} k^2 = \frac{m(m+1)(2m+1)}{6}}$$

THUS, WE GET

$$\sum_{k=500}^{1010} k = \sum_{k=1}^{1010} k - \sum_{k=1}^{499} k = \frac{1010 \cdot 1011}{2} - \frac{499 \cdot 500}{2}$$

$$= 510\,555 - 124\,750 = 385\,805 .$$

SIMILARLY,

$$\sum_{k=500}^{1010} k^2 = \sum_{k=1}^{1010} k^2 - \sum_{k=1}^{499} k^2$$

$$= \frac{1010 \cdot 1011 \cdot 2021}{2} - \frac{499 \cdot 500 \cdot 999}{6}$$

$$= 1\,031\,831\,655 - 41\,541\,750$$

$$= 990\,289\,905$$

WE THEREFORE HAVE

$$\sum_{k=500}^{1010} t_{2k} = 4 \cdot \sum_{k=500}^{1010} k^2 + 2 \cdot \sum_{k=500}^{1010} k$$

$$= 4 \cdot 990\,289\,905 + 2 \cdot 385\,805$$

$$= 3\,961\,931\,230$$

HENCE, IF $t_{1000} + t_{1002} + t_{1004} + \dots + t_{2020} + t_{2021} = S$ WE HAVE

$$S = \sum_{k=500}^{1010} t_{2k} + t_{2021}$$

$$= 3\,961\,931\,230 + \binom{2021+1}{2}$$

$$= 3\,961\,931\,230 + \frac{2021 \cdot 2022}{2}$$

$$= 3\,961\,931\,230 + 2\,043\,231$$

$$= 3\,963\,974\,461. \qquad \blacksquare$$

Extra exercise for practising

(1) FOR $n \geq 2$, PROVE THAT

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \dots + \binom{n}{2} = \binom{n+1}{3}.$$

(2) FROM (1) AND THE RELATION $m^2 = 2\binom{m}{2} + m$ FOR $m \geq 2$, DEDUCE THE FORMULA

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

(3) FIND THE EXACT VALUE OF

$$50 \cdot 51 + 51 \cdot 52 + 52 \cdot 53 + \dots + 198 \cdot 199 + 199 \cdot 200$$

BY USING THE PREVIOUS FORMULAS.

(4) FIND THE EXACT VALUE OF THE SUM

$$\sum_{k=2}^{n} \binom{2k+1}{2} \qquad \text{FOR EVERY } n \geq 2.$$