# MATHEMATICAL METHODS

LECTURE 3

## 1. Maps between sets

Consider two (non-empty) sets $A$ and $B$, and suppose that with each $x \in A$ there is associated exactly one element of $B$, which we denote by $f(x)$. Then $f$ is said to be a *function* from $A$ to $B$, or a *mapping* of $A$ into $B$, in symbols $f\colon A \to B$. The set $A$ is called the *domain* of $f$, where we also say that $f$ is defined on $A$, and the elements $f(x)$ are called the *values* of $f$. The set $\{f(x)\colon x \in A\}$ is called the *range* of $f$, and $\Gamma_f = \{(x, f(x))\colon x \in A\} \subseteq A \times B$ is the *graph* of $f$.

If $f\colon A \to B$ is a function and $E \subseteq A$, then $f(E) = \{f(x)\colon x \in E\}$ is called the *image* of $E$ under $f$. In this notation, the image of $A$ under $f$ is the range of $f$. Let $f\colon A \to B$ and $g\colon C \to D$ be two functions. If $f(A) \subseteq C$, then we can define the *function composition* of $g$ and $f$ as the map $g \circ f\colon A \to D$, which is defined as $(g \circ f)(x) = g(f(x))$ for all $x \in A$.

We have the following three fundamental types of functions:

(1) It is clear that $f(A) \subseteq B$. If $f(A) = B$, then we say that $f$ maps $A$ *onto* $B$. Equivalent notion is that $f$ is *surjective*. We have the following:

$$f\colon A \to B \text{ is surjective } \iff \forall y \in B.\ \exists x \in A.\ f(x) = y.$$

(2) If $E \subseteq B$, then we define the *inverse image* of $E$ under $f$ as the set $f^{-1}(E) = \{x \in A\colon f(x) \in E\}$. If $y \in B$, then we put $f^{-1}(y) = f^{-1}(\{y\})$. If for each $y \in B$, $f^{-1}(y)$ consists of at most one element of $A$, then $f$ is said to be a 1-1 (*one-to-one*) mapping of $A$ into $B$. Equivalent notion is also that $f$ is *injective*. We have the following:

$$f\colon A \to B \text{ is injective } \iff \forall x \in A.\ \forall y \in A.\ x \neq y \implies f(x) \neq f(y).$$

(3) $f\colon A \to B$ is said to be a *bijective* function, or that $A$ and $B$ can be put in *1-1 correspondence*, if the function is both surjective and injective. We have the following:

$$f\colon A \to B \text{ is bijective } \iff \forall y \in B.\ \exists! x \in A.\ f(x) = y.$$

**Remark 1.** Remember that two finite sets $A$ and $B$ have the same cardinality number if $|A| = |B|$, or in other words, they both have the same number of elements. For *infinite* sets we can now say that $A$ and $B$ have the same cardinality number, or briefly that $A$ and $B$ are *equivalent* (or *equipollent*), in symbols $A \sim B$, if there exists a bijective function $f\colon A \to B$.

If $f\colon A \to B$ is *bijective*, then there exists a unique function $g\colon B \to A$ with the property that $(g \circ f)(x) = x$ for every $x \in A$ and $(f \circ g)(y) = y$ for every $y \in B$. This function $g$ is called the *inverse* of $f$ and it is denoted as $f^{-1}$. We also have the following: *Let $f\colon A \to B$ and $g\colon B \to A$ be two functions such that $(g \circ f)(x) = x$ for every $x \in A$ and $(f \circ g)(y) = y$ for every $y \in B$. Then $f$ and $g$ are bijective functions.*

## 2. Number sets

2.1. **Natural numbers.** The simplest of all number sets is the *set of natural numbers* $\mathbb{N} = \{1, 2, 3, \ldots\}$. Formally, this set is defined by the *Peano axioms*:

(1) 1 is a natural number;
(2) Every natural number $n$ has exactly one successor $n^+$;
(3) 1 is not a successor of any natural number;
(4) Different natural numbers have different successors: $m \neq n \implies m^+ \neq n^+$;
(5) *Principle of mathematical induction*: Let $M \subseteq \mathbb{N}$ be a subset such that $1 \in M$ and $n \in M$ imply $n^+ \in M$. Then $M = \mathbb{N}$.

*Date*: 12. 10. 2026.

The principle of mathematical induction is an important tool for proving statements. The strategy is the following:

(1) *Induction basis*: We check that the statement holds for $n = 1$;
(2) *Induction hypothesis*: We assume that the statement holds for $n = k \in \mathbb{N}$;
(3) *Induction step*: We show that the truthiness of the statement for $n = k$ implies the truthiness of the statement for $n = k + 1$.

Then the statement will hold for all $n \in \mathbb{N}$ due to the 5th Peano axiom.

**Example 1.** Prove that $1+2+\cdots+n = n(n+1)/2$ holds for all $n \in \mathbb{N}$. Define $L(n) = 1+2+\cdots+n$ and $R(n) = n(n+1)/2$. We have $L(1) = 1$ and $R(1) = 1$, therefore $L(n) = R(n)$ holds for $n = 1$. Assume $L(k) = R(k)$ for $k \in \mathbb{N}$. Then $L(k+1) = 1 + 2 + \cdots + k + (k+1) = L(k) + k + 1$ and $R(k+1) = (k+1)(k+2)/2 = \big(k^2 + k + 2(k+1)\big)/2 = k(k+1)/2 + k + 1 = R(k) + k + 1$, so $L(k+1) = R(k+1)$. It then follows $L(n) = R(n)$ for all $n \in \mathbb{N}$.

2.2. **Integers.** The set of integers $\mathbb{Z}$ is defined as

$$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n \colon n \in \mathbb{N}\} = \{\ldots, -2, -1, 0, 1, 2, \ldots\},$$

where we define 0 as such number that $n + 0 = n$ holds for all $n \in \mathbb{N}$. Numbers in $\{-n \colon n \in \mathbb{N}\}$ are called *negative integers* and are defined in a way that $n + (-n) = 0$ for all $n \in \mathbb{N}$, while natural numbers are also known as *positive integers*. The introduction of integers has historical motivation: we want to solve $a + x = b$, where $a$ and $b$ are given natural numbers; however this equation is not always solvable in $\mathbb{N}$.

Both sets $\mathbb{N}$ and $\mathbb{Z}$ are clearly infinite. Although $\mathbb{N}$ is a proper subset of $\mathbb{Z}$, these two sets are in fact equivalent, i.e., $\mathbb{N} \sim \mathbb{Z}$, so they have the same cardinality. Consider the following function $f \colon \mathbb{N} \to \mathbb{Z}$, which is defined as

$$f(n) = \begin{cases} n/2, & 2 \mid n, \\ (1-n)/2, & 2 \nmid n. \end{cases}$$

A few of its values: $f(1) = 0$, $f(2) = 1$, $f(3) = -1$, $f(4) = 2$, $f(5) = -2$. In order to show that $f$ is bijective, take an arbitrary integer $m \in \mathbb{Z}$. We want to prove that then exists only one $n \in \mathbb{N}$ such that $f(n) = m$. According to the definition of $f$, the only possible options for the value of $n$ are $2m$ or $1 - 2m$. Note that $2m \neq 1 - 2m$ for any $m \in \mathbb{Z}$. If $m$ is positive, then we should have $n = 2m > 1$ since the other option in this case always produces a negative number. Similarly, if $m$ is negative, then we should have $n = 1 - 2m > 1$. If $m = 0$, the only possible option is $n = 1 - 2m = 1$.